COMBINATORIAL FORMULAE FOR GROTHENDIECK-DEMAZURE AND GROTHENDIECK POLYNOMIALS

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1. INTRODUCTION

This report will discuss evidence of the discovery of combinatorial rules for Grothendieck-Demazure and Grothendieck polynomials. Section 2 will discuss the case of the Grothendieck-Demazure polynomials. Section 3 will discuss the case of the Grothendieck polynomials.

2. GROTHENDIECK-DEMAZURE POLYNOMIALS

Suppose α is a composition such that $\alpha = (\alpha_1, \alpha_2, ...) \in \mathbb{N}^\infty$ and only a finite number of the α_i's are non-zero. For $f \in \mathbb{Z}[x_1, ..., x_n]$, let

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$$

where s_i acts on f by transposing x_i and x_{i+1} and let

$$\tilde{\pi}_i = \partial_i(x_i(1 - x_{i+1})f)$$

Then the *Grothendieck-Demazure polynomial* κ_α, which is attributed to A. Lascoux and M. P. Schützenberger, is defined as

$$\kappa_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ...$$

if $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq ...$, i.e. α is non-increasing, and

$$\kappa_\alpha = \tilde{\pi}_i \kappa_{\alpha s_i}$$

if $\alpha_i < \alpha_{i+1}$, where s_i acts on α by transposing the indices.

Example 2.1. Let $\alpha = (0, 2, 1)$. Then

$$\begin{aligned}\kappa_{(0,2,1)} &= \tilde{\pi}_1 \kappa_{\alpha s_1} \\ &= \tilde{\pi}_1(x_1^2 x_2 + x_1^2 x_3 - x_1^2 x_2 x_3) \\ &= x_1^2 x_2 + x_1 x_2^2 - x_1^2 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 - 2x_1^2 x_2 x_3 + x_2^2 x_3 - 2x_1 x_2^2 x_3 + x_1^2 x_2^2 x_3\end{aligned}$$

Let $\alpha = (\alpha_1, \alpha_2, ...)$ be a composition. Then $D(\alpha)$, which is the *key diagram of* α described in [ReiShi95], is an array of left-justified boxes, where there are α_i boxes in row i in the positions $[i, j]$ for $j = 1, 2, ..., \alpha_i$. Note that if $\alpha_i = 0$, then there are no boxes in row i of $D(\alpha)$.

Date: May 13, 2011.

Example 2.2. Let $\alpha = (1, 0, 3, 1, 2)$. Then

$$D(\alpha) = \begin{pmatrix} \square & & \\ & & \\ \square & \square & \square \\ \square & & \\ \square & \square & \end{pmatrix}$$

Suppose α is a composition and $\alpha_i \neq 0$. Let $[i, j] \in D(\alpha)$ such that $[i, j'] \notin D(\alpha)$ for $j' > j$, i.e. the box at $[i, j]$ is the rightmost box in the row. Let

$$M_{D(\alpha)}(i, j) = \{[i', j] \mid i' < i \text{ and } [i', j] \notin D(\alpha)\}$$

which is based on $M_D(i, j)$ in [Win03]. If $M_{D(\alpha)}(i, j) \neq \emptyset$, then a *Kohnert move*, which is described in [ReiShi95], consists of moving the box at $[i, j]$ to $[i', j]$, where i' is the greatest row number, i.e. the row closest to row i with an open space. Note that if $[i', j] \neq [i-1, j]$, then the Kohnert move is considered a *tunnel move* since the box at $[i, j]$ had to push or move through the boxes in positions $[i-1, j], [i-2, j], \ldots, [i'-1, j]$ (see [Win03]).

Definition 2.3. Given a composition α, let $\mathrm{Koh}(D(\alpha))$ be the set of all diagrams derived from $D(\alpha)$ by Kohnert moves.

Example 2.4. Let $\alpha = (0, 2, 1)$. Then

$$\mathrm{Koh}(D(0,2,1)) = \left\{ \begin{pmatrix} & \square \\ \square & \\ \square & \end{pmatrix}, \begin{pmatrix} \square & \\ \square & \square \\ & \end{pmatrix}, \begin{pmatrix} \square & \square \\ & \\ \square & \end{pmatrix}, \begin{pmatrix} \square & \square \\ \square & \\ & \end{pmatrix} \right\}$$

Definition 2.5. Let α be a composition and $[i, j] \in D(\alpha)$ such that $[i, j'] \notin D(\alpha)$ for $j' > j$, i.e. $[i, j]$ is the rightmost box in row i. Suppose $[i', j]$ is a Kohnert move of $[i, j]$. Then a *ghost-Kohnert move* consists of moving the box in $[i, j]$ to $[i', j]$ and leaving a copy or "ghost" of the original box at $[i, j]$, which is not allowed to move, i.e. no Kohnert or ghost-Kohnert moves allowed.

Definition 2.6. Given a composition α, let $\mathrm{GhostKoh}(D(\alpha))$ be the set of all diagrams derived from $D(\alpha)$ by ghost-Kohnert moves.

For notation in a diagram D, let $\boxdot$ denote a "ghost" box.

Example 2.7. Let $\alpha = (0, 2, 1)$. Then

$$\mathrm{GhostKoh}(D(0,2,1)) = \left\{ \begin{pmatrix} & \square \\ \square & \boxdot \\ \square & \end{pmatrix}, \begin{pmatrix} \square & \\ \square & \square \\ \boxdot & \end{pmatrix}, \begin{pmatrix} \square & \square \\ \boxdot & \\ \square & \end{pmatrix}, \begin{pmatrix} \square & \square \\ \square & \\ \boxdot & \end{pmatrix}, \begin{pmatrix} \square & \square \\ \square & \boxdot \\ & \end{pmatrix}, \begin{pmatrix} \square & \square \\ \square & \boxdot \\ \boxdot & \end{pmatrix} \right\}$$

Suppose α is a composition and D is a diagram such that $D = D(\alpha)$ or $D \in \mathrm{Koh}(D(\alpha))$. Then we can associate $x^D = x_1^{\beta_1} x_2^{\beta_2} \ldots$ to D, where $\beta_i = |D^{[i]}|$, i.e. β_i is the number of boxes in row i of D (see [Win03]).

Definition 2.8. Suppose α is a composition and $D \in \mathrm{GhostKoh}(D(\alpha))$. Then the monomial associated to D is $(-1)^g x^D = (-1)^g x_1^{\beta_1} x_2^{\beta_2} \ldots$, where g is the number of ghosts in D and $\beta_i = |D^{[i]}|$.

Conjecture 2.9. *Suppose α is a composition such that $\alpha = (\alpha_1, \alpha_2, ...) \in \mathbb{N}^\infty$ and only a finite number of the α_i's are non-zero. Then*

$$\kappa_\alpha = x^{D(\alpha)} + \sum_{D \,\in\, \mathrm{Koh}(D(\alpha))} x^D + \sum_{D \,\in\, \mathrm{GhostKoh}(D(\alpha))} (-1)^g x^D$$

where g is the number of ghosts in $D \in \mathrm{GhostKoh}(D(\alpha))$.

The above conjecture only deals with distinct diagrams because certain diagrams can be derived from a different combination of Kohnert and/or ghost-Kohnert moves.

Example 2.10. Let $\alpha = (0, 2, 1)$. From Example 2.1, we know that

$$\kappa_{(0,2,1)} = x_1^2x_2 + x_1x_2^2 - x_1^2x_2^2 + x_1^2x_3 + x_1x_2x_3 - 2x_1^2x_2x_3 + x_2^2x_3 - 2x_1x_2^2x_3 + x_1^2x_2^2x_3$$

Then $x^{D((0,2,1))} = x_2^2x_3$. From $\mathrm{Koh}(D(0, 2, 1))$ derived in Example 2.4 and $\mathrm{GhostKoh}(D(0, 2, 1))$ derived in Example 2.7, we have that

$$\sum_{D \,\in\, \mathrm{Koh}(D(0,2,1))} x^D = x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2$$

and

$$\begin{aligned}\sum_{D \,\in\, \mathrm{GhostKoh}(D(0,2,1))} (-1)^g x^D &= -x_1x_2^2x_3 - x_1x_2^2x_3 - x_1^2x_2x_3 - x_1^2x_2x_3 - x_1^2x_2^2 + (-1)^2x_1^2x_2^2x_3 \\ &= -2x_1x_2^2x_3 - 2x_1^2x_2x_3 - x_1^2x_2^2 + x_1^2x_2^2x_3\end{aligned}$$

Thus, it follows that

$$\kappa_{(0,2,1)} = x^{D((0,2,1))} + \sum_{D \,\in\, \mathrm{Koh}(D(0,2,1))} x^D + \sum_{D \,\in\, \mathrm{GhostKoh}(D(0,2,1))} (-1)^g x^D$$

We have computationally tested Conjecture 2.9 using our package `RulesGroth(Dem).m` in `Mathematica 7`. Given a composition `alpha` = {`alpha1`, ..., `alphak`}, `kappa`[`alpha`] returns the Grothendieck-Demazure polynomial of the composition by using the definition involving divided differences and `kapparule`[`alpha`] saves a list `diagrams`, which contains the diagram of the composition and all diagrams derived from Kohnert and ghost-Kohnert moves on the composition's diagram, and returns the polynomial that results from associating a monomial to each element in `diagrams`. Using those functions, we have tested Conjecture 2.9 for a variety of compositions $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$, i.e. different values of k and n where $n = \alpha_1 + \alpha_2 + ... + \alpha_k$, as well as skyline compositions and partitions. This testing has not produced counterexamples.

One conclusion that can be drawn from Conjecture 2.9 is the positivity of terms in κ_α.

Conjecture 2.11. *Given a composition α such that κ_α is defined, let d be the lowest degree of the terms in κ_α. Then the sign of any term $x_1^{k_1}x_2^{k_2}... \in \kappa_\alpha$ is $(-1)^{(k_1+k_2+...)-d}$.*

Example 2.12. Let $\alpha = (0, 2, 1)$. Then

$$\kappa_{(0,2,1)} = x_1^2x_2 + x_1x_2^2 - x_1^2x_2^2 + x_1^2x_3 + x_1x_2x_3 - 2x_1^2x_2x_3 + x_2^2x_3 - 2x_1x_2^2x_3 + x_1^2x_2^2x_3$$

For $\kappa_{(0,2,1)}$, the lowest degree of the terms is $d = 3$. The sign of the terms with degree d in $\kappa_{(0,2,1)}$, i.e. $x_1^2x_2$, $x_1x_2^2$, $x_1^2x_3$, $x_2^2x_3$, and $x_1x_2x_3$, is positive, which agrees with $(-1)^{d-d} = 1$. The sign of the terms with degree $d + 1 = 4$ in $\kappa_{(0,2,1)}$, i.e. $x_1^2x_2^2$, $x_1^2x_2x_3$, and $x_1x_2^2x_3$, is negative, which agrees with $(-1)^{(d+1)-d} = -1$. The sign of the coefficient of the term with degree $d + 2 = 5$, i.e. $x_1^2 x_2^2 x_3$, is positive, which agrees with $(-1)^{(d+2)-d} = 1$.

3. Grothendieck polynomials

Suppose $w \in S_n$. Let $X = (x_1, x_2, ...)$ and $Y = (y_1, y_2, ..)$ be sequences of commuting independent variables (see [LasSch82]). If $w = w_0$ is the permutation with the maximum number of inversions in S_n, i.e $w_0 = n(n-1)(n-2)...21$, then set the *Grothendieck polynomial*

$$\mathfrak{G}_{w_0}(X;Y) = \prod_{i+j \leq n} (x_i + y_j - x_i y_j).$$

Otherwise, $w \neq w_0$ so there exists a transposition $s_i = (i, i+1) \in S_n$ such that $l(ws_i) = l(w) + 1$. Then the *Grothendieck polynomial for* w is defined as

$$\mathfrak{G}_w = \pi_i(\mathfrak{G}_{ws_i})$$

where π_i is the following divided difference:

$$\pi_i(f) = \frac{(1 - x_{i+1}) f(x_1, ..., x_i, x_{i+1}, ..., x_n) - (1 - x_i) f(x_1, ..., x_{i+1}, x_i, ..., x_n)}{x_i - x_{i+1}}$$

For this paper, we are considering single Grothendieck polynomials. Thus, for $w \in S_n$, we calculate $\mathfrak{G}_w(X;Y)$ and then set $Y = (0, 0, ..., 0)$, i.e. $y_i = 0$ for all i.

Example 3.1. Let $w = 2143 \in S_4$. Then $\mathfrak{G}_{2143} = x_1^2 + x_1x_2 - x_1^2x_2 + x_1x_3 - x_1^2x_3 - x_1x_2x_3 + x_1^2x_2x_3$.

The *diagram of a permutation* $w \in S_n$, denoted by $D(w)$, is a finite collection of boxes with vertices in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ (see [Win03]). For $D(w)$, there are boxes in positions $\{[i, j] \mid 1 \leq i, j \leq n\}$ except where cancellation occurs from the "hooks" given by

$$\{[w(j), j'] \mid j' \geq j\} \cup \{[i', j] \mid i' \geq w(j)\}$$

Example 3.2. Let $w = 51432 \in S_5$. Then

$$D(w) = \begin{pmatrix} \square & & & \\ \square & & \square & \\ \square & & \square & \square \\ \square & & & \end{pmatrix}$$

Let $w \in S_n$ and $[i, j] \in D(w)$ such that $[i', j] \notin D(w)$ for $i' > i$, i.e. the box at $[i, j]$ is the highest box in the column. Let

$$M_{D(w)}(i, j) = \{[i, j'] \mid j' < j \text{ and } [i, j'] \notin D(w)\}$$

which Winkel used in [Win03]. If $M_{D(w)}(i, j) \neq \emptyset$, then a *Kohnert move* consists of moving the box at $[i, j]$ to $[i, j']$, where j' is the greatest column number, i.e. the column closest to column j with an open space. Note that if $[i, j'] \neq [i, j-1]$, then the Kohnert move is considered a *tunnel move* since the box at $[i, j]$ had to push or move through the boxes in positions $[i, j-1], [i, j-2], ..., [i, j'-1]$ (see [Win03]).

Definition 3.3. Given a permutation $w \in S_n$, let $\mathrm{Koh}(D(w))$ be the set of all diagrams derived from $D(w)$ by Kohnert moves.

Example 3.4. Let $w = 2143 \in S_4$. Then

$$\mathrm{Koh}(D(2143)) = \left\{ \begin{pmatrix} & \square & \\ & & \\ \square & & \end{pmatrix}, \begin{pmatrix} \square & & \\ & & \\ \square & & \end{pmatrix} \right\}$$

Definition 3.5. Let $w \in S_n$ and $[i,j] \in D(w)$ such that $[i',j] \notin D(w)$ for $i' > i$, i.e. $[i,j]$ is the highest box in column j. Suppose $[i,j']$ is a Kohnert move of $[i,j]$. Then a *ghost-Kohnert move* consists of moving the box in $[i,j]$ to $[i,j']$ and leaving a copy or "ghost" of the original box at $[i,j]$, which is not allowed to move, i.e. no Kohnert or ghost-Kohnert moves allowed.

Definition 3.6. Given $w \in S_n$, let $\mathrm{GhostKoh}(D(w))$ be the set of all diagrams derived from $D(w)$ by ghost-Kohnert moves.

We still use the same notation $\boxdot$ to denote a "ghost" box.

Example 3.7. Let $w = 2143 \in S_4$. Then

$$\mathrm{GhostKoh}(D(2143)) = \left\{ \begin{pmatrix} & \square & \boxdot \\ & & \\ \square & & \end{pmatrix}, \begin{pmatrix} \square & \boxdot & \\ & & \\ \square & & \end{pmatrix}, \begin{pmatrix} \square & & \boxdot \\ & & \\ \square & & \end{pmatrix}, \begin{pmatrix} \square & \boxdot & \boxdot \\ & & \\ \square & & \end{pmatrix} \right\}$$

Given $w \in S_n$, the association of the monomial $x^D = x_1^{\beta_1} x_2^{\beta_2} \ldots$ to a diagram D, where $\beta_j = |D^{[j]}|$ is the number of boxes in column j of D, is applicable for D such that $D = D(w)$ or $D \in \mathrm{Koh}(D(w))$ (see Win[03]).

Definition 3.8. Suppose $w \in S_n$ and $D \in \mathrm{GhostKoh}(D(w))$. Then the monomial associated to D is $(-1)^g x^D = (-1)^g x_1^{\beta_1} x_2^{\beta_2} \ldots$, where g is the number of ghosts in D and $\beta_j = |D^{[j]}|$, i.e. β_j is the number of boxes in column j of D.

Conjecture 3.9. *Let $w \in S_n$. Then the Grothdendieck polynomial $\mathfrak{G}_w$ is given by*

$$\mathfrak{G}_w = x^{D(w)} + \sum_{D \,\in\, \mathrm{Koh}(D(w))} x^D + \sum_{D \,\in\, \mathrm{GhostKoh}(D(w))} (-1)^g x^D$$

where g is the number of ghosts in $D \in \mathrm{GhostKoh}(D(w))$.

The above conjecture only deals with distinct diagrams because certain diagrams can be derived from a different combination of Kohnert and/or ghost-Kohnert moves. Note that Conjecture 3.9 gives a generating series for the Grothendieck polynomials.

Example 3.10. Let $w = 2143 \in S_4$. From Example 3.1, we know that

$$\mathfrak{G}_{2143} = x_1^2 + x_1x_2 - x_1^2x_2 + x_1x_3 - x_1^2x_3 - x_1x_2x_3 + x_1^2x_2x_3$$

Then $x^{D(2143)} = x_1x_3$. From $\mathrm{Koh}(D(2143))$ derived in Example 3.4 and $\mathrm{GhostKoh}(D(2143))$ derived in Example 3.7, we have that

$$\sum_{D \,\in\, \mathrm{Koh}(D(2143))} x^D = x_1x_2 + x_1^2$$

and

$$\sum_{D \in \text{GhostKoh}(D(2143))} (-1)^g x^D = -x_1x_2x_3 - x_1^2x_2 - x_1^2x_3 + (-1)^2x_1^2x_2x_3$$

Thus, it follows that

$$\mathfrak{G}_{2143} = x^{D(2143)} + \sum_{D \in \text{Koh}(D(2143))} x^D + \sum_{D \in \text{GhostKoh}(D(2143))} (-1)^g x^D$$

We have computationally tested Conjecture 3.9 using our package `RulesGroth(Dem).m` in `Mathematica 7`. Given a permutation `w` $=$ {`w1`, ..., `wk`}, `grothendieck`[`w`] returns the Grothendieck polynomial of the permutation by using the definition involving divided differences and `grothrule`[`w`] saves a list `diagrams`, which contains the diagram of the permutation and all diagrams derived from Kohnert and ghost-Kohnert moves on the permutation's diagram, and returns the polynomial that results from associating a monomial to each element in `diagrams`. Using those functions, we have tested Conjecture 3.9 for a variety of permutations in S_n for different n. This testing has not produced counterexamples.

Another conjectured combinatorial rule for the Grothendieck polynomials involves swapped permutations.

Definition 3.11. Let $w \in S_n$ and $k = w^{-1}(1)$. Then let $s_{k,i}$ denote the transposition that acts on the indices k and i of w.

Example 3.12. Let $w = 1432 \in S_4$. Then $w' = ws_{1,4} = 2431$.

Given $w \in S_n$, the set of the indices of all possible singles swaps in w, which is defined by Winkel in [Win03], is

$$J^{>k}(w) = \{j \mid k < j, w(k) < w(j), \text{ and } |\{v \mid k < v < j, w(k) < w(v) < w(j)\}| = 0\}$$

Definition 3.13. Suppose $w \in S_n$. Then a swapped permutation of w is

$$w' = ws_{k,i_1}s_{k,i_2}...s_{k,i_m}$$

where $i_j \in J^{>k}(w)$ such that $i_1 > i_2 > ... > i_m$ and $m \geq 1$.

Example 3.14. Let $w = 1432 \in S_4$. Then $w' = ws_{1,4}s_{1,3}s_{1,2} = 4321$.

Definition 3.15. Given $w \in S_n$, let $\text{swap}(w)$ denote the set of all swapped permutations of w.

Example 3.16. Let $w = 1432 \in S_4$. Then

$$\text{swap}(1432) = \{2431, 3412, 4132, 3421, 4231, 4312, 4321\}$$

Conjecture 3.17. *Let $w \in S_n$ and $k = w^{-1}(1)$. Then*

$$\mathfrak{G}_w = \frac{1}{x_k} \sum_{w' \in \text{swap}(w)} (-1)^s \mathfrak{G}_{w'}$$

where s is the number of swaps.

An important part of the proof needed for Conjecture 3.17 is a bijection between the set

$$\{D \mid D = D(w), D \in \mathrm{Koh}(D(w)), \text{ or } D \in \mathrm{GhostKoh}(D(w))\}$$

for $w \in S_n$ and the set

$$\{D \mid w' \in \mathrm{swap}(w), \text{ and } D = D(w'), D \in \mathrm{Koh}(D(w')), \text{ or } D \in \mathrm{GhostKoh}(D(w'))\}$$

Claim 3.18. *Suppose $w \in S_n$ such that $w(n) \neq 1$. Let $k = w^{-1}(1)$ and $w' \in \mathrm{swap}(w)$. For any diagram D such that $D = D(w'), D \in \mathrm{Koh}(D(w'))$, or $D \in \mathrm{GhostKoh}(D(w'))$, there is a box at $(1, k)$.*

We conclude the paper by examining Grothendieck polynomials in terms of Grothendieck-Demazure polynomials.

Conjecture 3.19. *A Grothdendieck polynomial can be written as an expansion of Grothendieck-Demazure polynomials κ_α's.*

Example 3.20. Let $w = 325164 \in S_6$. Then

$$\begin{aligned}\mathfrak{G}_{325164} = {} & \kappa_{(2,1,2,0,1)} + \kappa_{(2,2,2,0,0)} - \kappa_{(2,2,2,0,1)} + \kappa_{(3,1,1,0,1)} + \kappa_{(3,1,2,0,0)} - 2\kappa_{(3,1,2,0,1)} \\ & - \kappa_{(3,2,2,0,0)} + \kappa_{(3,2,2,0,1)}\end{aligned}$$

We have computationally tested Conjecture 3.19 using our package `RulesGroth(Dem).m` in `Mathematica 7`. Given a permutation `w` $= \{$`w1`$, ...,$ `wk`$\}$, `grothendieckexpansion[w]` returns the Grothendieck polynomial of the permutation expressed in terms of Grothendieck-Demazure polynomials using the following algorithm provided by Alexander Yong:

```
Let A = grothendieck[w] and finalanswer = 0
While (A ≠ 0)
    c_α = the coefficient of the reverse lexicographic largest c_α x^α
        among the lowest degree terms in A
    A = A − c_α*kappa[alpha]
    finalanswer = finalanswer + c_α*kappa[alpha]
end While
Return[finalanswer]
```

Using this function and `grothendieck[w]`, we have tested Conjecture 3.19 for a variety of permutations in S_n for different n. This testing has not produced counterexamples.

Conjecture 3.21. *Consider a Grothendieck polynomial $\mathfrak{G}_w$ and its expansion in terms of Grothendieck-Demazure polynomials κ_α's. Let $|\alpha| = \alpha_1 + \alpha_2 + ...$ be the weight of a composition. For a κ_α in the expansion, the sign of the coefficient of κ_α is $(-1)^{|\alpha|-|D(w)|}$, where $|D(w)|$ is the number of boxes in $D(w)$.*

We have computationally tested Conjecture 3.21 using our package `RulesGroth(Dem).m` in `Mathematica 7`. Given a permutation `w` $= \{$`w1`$, ...,$ `wk`$\}$, `coefficientconj[w]` returns `True` only if the sign of each Grothendieck-Demazure polynomial in the calculated expansion of the Grothendieck polynomial of `w` follows the rule in Conjecture 3.21. Using this function, we have tested Conjecture 3.21 for a variety of permutations in S_n for different n. This testing has not produced counterexamples. Although it seemed like the coefficients of the κ_α's were either -1 or 1, this is not the case based on testing the conjecture for small examples, i.e. there is multiplicity in the expansion as seen in Example 3.20.

References

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[Win03] R. Winkel, *A derivation of Kohnert's algorithm from Monk's rule*, Sem. Loth. Comb., B48f (2003).