AN EFFICIENT ALGORITHM FOR DECIDING VANISHING OF SCHUBERT POLYNOMIAL COEFFICIENTS

ANSHUL ADVE, COLLEEN ROBICHAUX, AND ALEXANDER YONG

ABSTRACT. *Schubert polynomials* form a basis of all polynomials and appear in the study of cohomology rings of flag manifolds. The vanishing problem for Schubert polynomials asks if a coefficient of a Schubert polynomial is zero. We give a tableau criterion to solve this problem, from which we deduce the first polynomial time algorithm. These results are obtained from new characterizations of the *Schubitope*, a generalization of the permutahedron defined for any subset of the $n \times n$ grid. In contrast, we show that computing these coefficients explicitly is #P-complete.

1. INTRODUCTION

Schubert polynomials form a linear basis of all polynomials $\mathbb{Z}[x_1, x_2, x_3, \ldots]$. They were introduced by A. Lascoux–M.-P. Schützenberger [7] to study the cohomology ring of the flag manifold. These polynomials represent the Schubert classes under the Borel isomorphism. A reference is the textbook [4].

If $w_0 = n\ n-1\ \cdots 2\ 1$ is the longest length permutation in S_n, then

$$\mathfrak{S}_{w_0}(x_1, \ldots, x_n) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Otherwise, $w \neq w_0$ and there exists i such that $w(i) < w(i+1)$. Then one sets

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \partial_i \mathfrak{S}_{ws_i}(x_1, \ldots, x_n),$$

where s_i is the transposition swapping i and $i+1$ and

$$\partial_i f := \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

Since ∂_i satisfies

$$\partial_i \partial_j = \partial_j \partial_i \text{ for } |i-j| > 1, \text{ and } \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1},$$

the above description of $\mathfrak{S}_w$ is well-defined. In addition, under the inclusion $\iota : S_n \hookrightarrow S_{n+1}$ defined by $w(1) \cdots w(n) \mapsto w(1)\ \cdots w(n)\ n+1$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. Thus one unambiguously refers to $\mathfrak{S}_w$ for each $w \in S_\infty = \bigcup_{n \geq 1} S_n$.

The *graph* $G(w)$ of a permutation $w \in S_n$ is the $n \times n$ grid, with a $\bullet$ placed in position $(i, w(i))$ (in matrix coordinates). The *Rothe diagram* of w is given by

$$D(w) = \{(i,j) : 1 \leq i, j \leq n, j < w(i), i < w^{-1}(j)\}.$$

This is pictorially described with rays that strike out boxes south and east of each $\bullet$ in $G(w)$. $D(w)$ are the remaining boxes.

The *code* of w, denoted $\mathsf{code}(w)$ is the vector $(c_1, c_2, \ldots, c_L)$ where c_i is the number of boxes in the i-th row of $D(w)$ and L indexes the southmost row with a positive number of boxes. To each $w \in S_\infty$ there is a unique associated code; see [8, Proposition 2.1.2].

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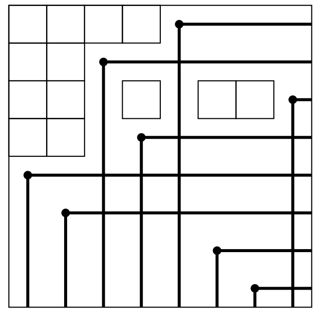
Example 1.1. If $w = 53841267 \in S_8$ (in one line notation) then $D(w)$ is depicted by:



Here, $\mathsf{code}(w) = (4, 2, 5, 2)$.

Consider the monomial expansion

$$\mathfrak{S}_w = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_{\alpha,w} x^\alpha.$$

Now, $c_{\alpha,w} = 0$ unless $\alpha_i = 0$ for $i > L$, and moreover, $c_{\alpha,w} \in \mathbb{Z}_{\geq 0}$. Let $\mathsf{Schubert}$ be the problem of deciding $c_{\alpha,w} \neq 0$, as measured in the input size of α and w (under the assumption that arithmetic operations take constant time). The `INPUT` is $\mathsf{code} = (c_1, \ldots c_L) \in \mathbb{Z}_{\geq 0}^L$ with $c_L > 0$ and $\alpha \in \mathbb{Z}_{\geq 0}^L$. $\mathsf{Schubert}$ returns `YES` if $c_{\alpha,w} > 0$ and `NO` otherwise.

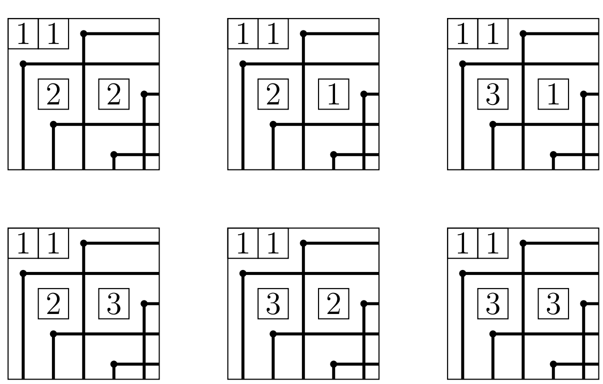
Theorem 1.2. $\mathsf{Schubert} \in \mathsf{P}$.

We prove Theorem 1.2 using another result. Fix $n \in \mathbb{Z}_{>0}$ and let $D \subseteq [n]^2$. We call D a *diagram* and visualize D as a subset of an $n \times n$ grid of boxes, oriented so that $(r, c) \in [n]^2$ represents the box in the rth row from the top and the cth column from the left. Let $\mathsf{PerfectTab}(D, \alpha)$ be the fillings of D with α_k many k's, where entries in each column are distinct, any entry in row i is $\leq i$, and each box contains exactly one entry. Let $\mathsf{PerfectTab}_{\downarrow}(D, \alpha) \subseteq \mathsf{PerfectTab}(D, \alpha)$ be fillings where entries in each column increase from top to bottom.

Theorem 1.3. $c_{\alpha,w} > 0 \iff \mathsf{PerfectTab}(D(w), \alpha) \neq \emptyset \iff \mathsf{PerfectTab}_{\downarrow}(D(w), \alpha) \neq \emptyset$

In general $\#\mathsf{PerfectTab}(D(w), \alpha) \neq c_{\alpha,w}$ but rather $\#\mathsf{PerfectTab}(D(w), \alpha) \geq c_{\alpha,w}$ (cf. [3]).

Example 1.4. Here are the tableaux in $\bigcup_\alpha \mathsf{PerfectTab}_{\downarrow}(D(31524), \alpha)$:



Hence, for instance, $c_{(2,1,1),31524} > 0$ but $c_{(4),31524} = 0$.

To prove Theorems 1.2 and 1.3 we establish results about the *Schubitope* introduced in [9]. This polytope $\mathcal{S}_D$ is defined with a halfspace description for any $D \subseteq [n]^2$. We prove (Theorem 2.13) that a lattice point α is in $\mathcal{S}_D$ if and only if $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ where D is any diagram.

We then introduce the *indicator polytope* $\mathcal{P}(D, \alpha)$ whose lattice points $\mathcal{P}(D, \alpha)_{\mathbb{Z}}$ are in bijection with $\mathsf{PerfectTab}(D, \alpha)$. We prove that $\mathcal{P}(D, \alpha) \neq \emptyset \iff \mathcal{P}(D, \alpha)_{\mathbb{Z}} \neq \emptyset$ (Theorem 2.27). Thus determining $\mathcal{P}(D, \alpha)_{\mathbb{Z}} \neq \emptyset$ (and equivalently $\alpha \in \mathcal{S}_D$) is in P using L. Khachiyan's ellipsoid method for linear programming, see [12]. We give two proofs of Theorem 2.27. The first shows $\mathcal{P}(D, \alpha)$ is totally unimodular. Hence $\mathcal{P}(D, \alpha) \neq \emptyset$ implies $\mathcal{P}(D, \alpha)$ has integral vertices. Our second proof obviates total unimodularity and is potentially adaptable to problems lacking that property. However, only the high-level structure of the second proof is easily generalizable — the rest is necessarily *ad hoc*.

For the special case of Rothe diagrams $D = D(w)$, using results of A. Fink-K. Mészáros-A. St. Dizier [2, Corollary 12 and Theorem 14] conjectured in [9, Conjectures 5.1 and 5.13],

$$\alpha \in \mathcal{S}_{D(w)} \iff c_{\alpha,w} > 0. \tag{1}$$

This, combined with our results on the Schubitope, proves Theorems 1.2 and 1.3.

The class $\#\mathsf{P}$ in L. Valiant's complexity theory of counting problems are those that count the number of accepting paths of a nondeterministic Turing machine running in polynomial time. A problem $\mathcal{P} \in \#\mathsf{P}$ is *complete* if for any problem $\mathcal{Q} \in \#\mathsf{P}$ there exists a polynomial-time counting reduction from $\mathcal{Q}$ to $\mathcal{P}$. These are the hardest of the problems in $\#\mathsf{P}$. There does not exist a polynomial time algorithm for such problems unless $\mathsf{P} = \mathsf{NP}$.

In contrast with Theorem 1.2, we prove:

Theorem 1.5. *Counting* $c_{\alpha,w}$ *is* $\#\mathsf{P}$-complete.

Given $\{c_{\alpha,w} \in \mathbb{Z}_{\geq 0}\}$ it is standard to ask for a counting rule for $c_{\alpha,w}$. A complexity motivation is an *appropriate* rule that establishes a counting problem is in $\#\mathsf{P}$ with respect to given input (length). The rule of [1] establishes that counting $c_{\alpha,w}$ is in $\#\mathsf{P}$ if the input is (w, α) but not if the input is $(\mathsf{code}(w), \alpha)$. For the latter input assumption, we use the transition algorithm of [6] and its *graphical* reformulation from [5]. This allows us to give a polynomial time counting reduction to the $\#\mathsf{P}$-complete problem of counting Kostka coefficients [10], (see Section 5).

2. The Schubitope

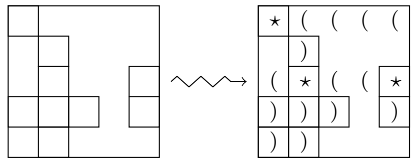
Consider a diagram $D \subseteq [n]^2$. Given $S \subseteq [n]$ and a column $c \in [n]$, construct a string denoted $\mathsf{word}_{c,S}(D)$ by reading column c from top to bottom and recording

- (if $(r, c) \notin D$ and $r \in S$,
-) if $(r, c) \in D$ and $r \notin S$, and
- $\star$ if $(r, c) \in D$ and $r \in S$.

Let $\theta_D^c(S) = \#\{\star\text{'s in } \mathsf{word}_{c,S}(D)\} + \#\{\text{paired ()'s in } \mathsf{word}_{c,S}(D)\}$ and

$$\theta_D(S) = \sum_{c=1}^{n} \theta_D^c(S).$$

Example 2.1. In the diagram D below, we labelled the corresponding strings for $\mathsf{word}_{c,S}(D)$ for $S = \{1, 3\}$. For instance, we see $\mathsf{word}_{5,\{1,3\}}(D) = (\star)$.



The *Schubitope* $\mathcal{S}_D$, as defined in [9], is the polytope

$$\left\{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n_{\geq 0} : \alpha_1+\cdots+\alpha_n = \#D \text{ and } \sum_{i\in S}\alpha_i \leq \theta_D(S) \text{ for all } S\subseteq[n]\right\}. \tag{2}$$

2.1. **Characterizations via tableaux.** A *tableau* of *shape* D is a map

$$\tau : D \to [n]\cup\{\circ\},$$

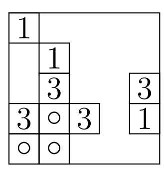
where $\tau(r,c)=\circ$ indicates that the box (r,c) is unlabelled. Let $\mathsf{Tab}(D)$ denote the set of such tableaux.

It will be useful to reformulate the original definition of $\theta_D(S)$ into the language of tableaux. Given $S\subseteq[n]$, define $\pi_{D,S}\in\mathsf{Tab}(D)$ by

$$\pi_{D,S}(r,c) = \begin{cases} r & \text{if } (r,c) \text{ contributes a ``}\star\text{'' to } \mathsf{word}_{c,S}(D),\\ s & \text{if } (r,c) \text{ contributes a ``)'' to } \mathsf{word}_{c,S}(D) \text{ which is} \\ & \text{paired with an ``('' from } (s,c),\\ \circ & \text{otherwise.}\end{cases} \tag{3}$$

In (3) and throughout, we pair by the standard "inside-out" convention.

Example 2.2. Continuing Example 2.1, below is $\pi_{D,\{1,3\}}(D)$

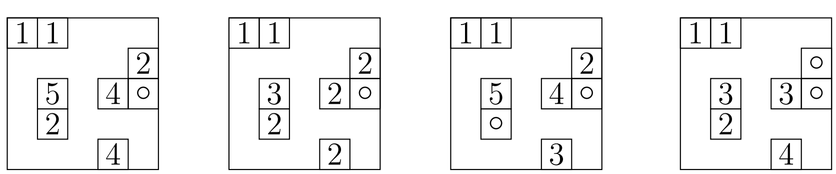


Proposition 2.3. *For all $D\subseteq[n]^2$ and $S\subseteq[n]$, we have $\theta_D(S)=\#\pi^{-1}_{D,S}(S)$.*

Proof. $\pi_{D,S}(r,c)\in S$ if and only if (r,c) falls into one of the first two cases in (3). ☐

Say $\tau\in\mathsf{Tab}(D)$ is *flagged* if $\tau(r,c)\leq r$ whenever $\tau(r,c)\neq\circ$. It is *column-injective* if $\tau(r,c)\neq\tau(r',c)$ whenever $r\neq r'$ and $\tau(r,c)\neq\circ$. Let $\mathsf{FCITab}(D)\subseteq\mathsf{Tab}(D)$ be the set of tableaux of shape D which are flagged and column-injective.

Example 2.4. Of the tableaux of shape D below, only the second and fourth are flagged, and only the third and fourth are column-injective.



Proposition 2.5. $\pi_{D,S}\in\mathsf{FCITab}(D)$ *for all $D\subseteq[n]^2$ and $S\subseteq[n]$.*

Proof. This is immediate from (3). ☐

A simple consequence of being flagged and column-injective is the following.

Proposition 2.6. *Let $\tau \in \mathsf{FCITab}(D)$. Then for all $(r,c) \in [n]^2$ and $S \subseteq [n]$, we have*

$$\#\{(i,c) \in \tau^{-1}(S) : i < r\} \leq \#\{i \in S : i \leq r\}, \tag{4}$$

with strict inequality whenever $(r,c) \in \tau^{-1}(S)$.

Proof. The map $(i,c) \mapsto \tau(i,c)$ from $\{(i,c) \in \tau^{-1}(S) : i \leq r\}$ to $\{i \in S : i \leq r\}$ is well-defined since τ is flagged. It is injective since τ is column-injective. Thus (4) holds, and

$$\#\{(i,c) \in \tau^{-1}(S) : i < r\} < \#\{(i,c) \in \tau^{-1}(S) : i \leq r\} \leq \#\{i \in S : i \leq r\}$$

whenever $(r,c) \in \tau^{-1}(S)$, establishing the strict inequality assertion. ☐

In fact, a stronger assertion holds when $\tau = \pi_{D,S}$.

Proposition 2.7. *If $(r,c) \in D \subseteq [n]^2$ and $S \subseteq [n]$, then*

$$(r,c) \in \pi_{D,S}^{-1}(S) \iff \#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} < \#\{i \in S : i \leq r\}.$$

Proof. ($\Rightarrow$) This direction follows from Propositions 2.5 and 2.6.
($\Leftarrow$) If $r \in S$, then (r,c) contributes a "$\star$" to $\mathsf{word}_{c,S}(D)$, so $\pi_{D,S}(r,c) = r \in S$, as desired. Thus we assume $r \notin S$. The hypothesis combined with this assumption says

$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} < \#\{i \in S : i \leq r\} = \#\{i \in S : i < r\}.$$

Thus, there is a maximal $s \in S$ with $s < r$ such that $\pi_{D,S}(r',c) \neq s$ whenever $r' < r$. If $(s,c) \in D$, then (s,c) contributes a "$\star$" to $\mathsf{word}_{c,S}(D)$, so $\pi_{D,S}(s,c) = s$, contradicting our choice of s. Therefore, (s,c) contributes an "(" to $\mathsf{word}_{c,S}(D)$. If this "(" is paired by a ")" contributed by $(r',c) \in D$ with $r' < r$, then $\pi_{D,S}(r',c) = s$, again a contradiction. Thus, this "(" pairs the ")" from (r,c), so $\pi_{D,S}(r,c) = s \in S$. Hence, $(r,c) \in \pi_{D,S}^{-1}(S)$ as desired. ☐

The previous two propositions combined assert that $\{(r,c) \in \pi_{D,S}^{-1}(S)\}$ is characterized by greedy selection as one moves down each column c. The next proposition shows that this greedy algorithm maximizes $\#\tau^{-1}(S)$ among all $\tau \in \mathsf{FCITab}(D)$.

Proposition 2.8. *Let $D \subseteq [n]^2$ and $S \subseteq [n]$. Then $\#\pi_{D,S}^{-1}(S) \geq \#\tau^{-1}(S)$ for all $\tau \in \mathsf{FCITab}(D)$.*

Proof. If not, then there exist $\tau \in \mathsf{FCITab}(D)$ and $(r,c) \in [n]^2$ satisfying

$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i \leq r\} < \#\{(i,c) \in \tau^{-1}(S) : i \leq r\}$$

and we can choose these such that r is minimized. Then because r is minimal,

$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} = \#\{(i,c) \in \tau^{-1}(S) : i < r\}$$

and $(r,c) \in \tau^{-1}(S) \smallsetminus \pi_{D,S}^{-1}(S)$, so in particular $(r,c) \in D$. Thus Proposition 2.6 implies

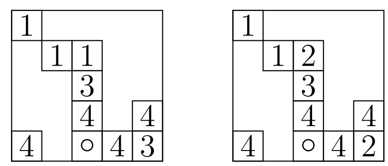
$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} = \#\{(i,c) \in \tau^{-1}(S) : i < r\} < \#\{i \in S : i \leq r\}.$$

But then we must have $(r,c) \in \pi_{D,S}^{-1}(S)$ by Proposition 2.7, a contradiction. ☐

If τ has shape a subset of $[n]^2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{\geq 0}$, say τ *exhausts* α *over* S if

$$\sum_{i \in S} \alpha_i \leq \#\tau^{-1}(S).$$

Example 2.9. Only the left tableau below exhausts $\alpha = (3, 2, 2, 4)$ over $S = \{1, 3\}$.



Theorem 2.10. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ with $\alpha_1 + \cdots + \alpha_n = \#D$. Then $\alpha \in \mathcal{S}_D$ if and only if for each $S \subseteq [n]$, there exists $\tau_{D,S} \in \mathsf{FClTab}(D)$ which exhausts α over S.*

Proof of Theorem 2.10. ($\Rightarrow$) The inequalities in (2) combined with Proposition 2.3 imply

$$\sum_{i \in S} \alpha_i \leq \theta_D(S) = \#\pi^{-1}_{D,S}(S).$$

Thus, $\tau_{D,S} := \pi_{D,S}$ exhausts α over S.

($\Leftarrow$) By Propositions 2.8 and 2.3,

$$\sum_{i \in S} \alpha_i \leq \#\tau^{-1}_{D,S}(S) \leq \#\pi^{-1}_{D,S}(S) = \theta_D(S),$$

so the inequalities in (2) hold. $\square$

Remark 2.11. The proof of ($\Rightarrow$) shows that we can take $\tau_{D,S} = \pi_{D,S}$ in Theorem 2.10.

It would be nice if $\tau_{D,S}$ did not depend on S, i.e., if some τ_D exhausted α over all $S \subseteq [n]$, so we could take $\tau_{D,S} = \tau_D$ in Theorem 2.10. Indeed, this is shown in Theorem 2.13.

Say $\tau \in \mathsf{Tab}(D)$ has *content* α if $\#\tau^{-1}(\{i\}) = \alpha_i$ for each $i \in [n]$. Let $\mathsf{Tab}(D, \alpha)$ and $\mathsf{FClTab}(D, \alpha)$ be the subsets of $\mathsf{Tab}(D)$ and $\mathsf{FClTab}(D)$, respectively, of those tableaux which have content α. In addition, call a tableau $\tau \in \mathsf{Tab}(D)$ *perfect* if $\tau \in \mathsf{FClTab}(D)$, and if no boxes are left unlabelled, i.e., $\tau^{-1}(\{\circ\}) = \emptyset$. Thus, the set of perfect tableaux of content α is precisely $\mathsf{PerfectTab}(D, \alpha) \subseteq \mathsf{FClTab}(D, \alpha)$ introduced in Section 1.

Proposition 2.12. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathsf{FClTab}(D, \alpha) \neq \emptyset$.*

Proof. ($\Rightarrow$) Let $\tau \in \mathsf{PerfectTab}(D, \alpha)$. Then $\tau \in \mathsf{FClTab}(D, \alpha)$, and since τ has content α and satisfies $\tau^{-1}(\{\circ\}) = \emptyset$,

$$\alpha_1 + \cdots + \alpha_n = \#\tau^{-1}(\{1\}) + \cdots + \#\tau^{-1}(\{n\}) = \#D.$$

($\Leftarrow$) Let $\tau \in \mathsf{FClTab}(D, \alpha)$. Then since τ has content α,

$$\#\tau^{-1}(\{\circ\}) = \#D - \#\tau^{-1}(\{1\}) - \cdots - \#\tau^{-1}(\{n\}) = \#D - \alpha_1 - \cdots - \alpha_n = 0.$$

Thus, $\tau \in \mathsf{PerfectTab}(D, \alpha)$. $\square$

Theorem 2.13. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\alpha \in \mathcal{S}_D$ if and only if $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$.*

The proof will require a lemma regarding tableaux of the form $\tau = \pi_{D,S}$.

Lemma 2.14. *Let $D \subseteq [n]^2$, and $S, T \subseteq [n]$ be disjoint. Set*

$$\tilde{D} = D \setminus \pi^{-1}_{D,S}(S) \text{ and } U = S \cup T.$$

Then

$$\pi^{-1}_{D,U}(U) = \pi^{-1}_{D,S}(S) \cup \pi^{-1}_{\tilde{D},T}(T).$$

Proof. Let $(r, c) \in D$, and assume by induction on r that

$$(i, c) \in \pi^{-1}_{D,U}(U) \iff (i, c) \in \pi^{-1}_{D,S}(S) \cup \pi^{-1}_{\tilde{D},T}(T) \tag{5}$$

whenever $i < r$. This clearly holds in the base case $r = 1$. By Proposition 2.7, $(r, c) \in \pi^{-1}_{D,U}(U)$ if and only if

$$\#\{(i, c) \in \pi^{-1}_{D,U}(U) : i < r\} < \#\{i \in U : i \leq r\}. \tag{6}$$

By (5) and the fact that

$$\pi^{-1}_{D,S}(S) \cap \tilde{D} = \emptyset = S \cap T,$$

(6) is equivalent to

$$\#\{(i, c) \in \pi^{-1}_{D,S}(S) : i < r\} + \#\{(i, c) \in \pi^{-1}_{\tilde{D},T}(T) : i < r\} < \#\{i \in S : i \leq r\} + \#\{i \in T : i \leq r\}.$$

By applying Proposition 2.6 twice, we see that this holds if and only if at least one of (i) and (ii) below hold.

(i) $\#\{(i, c) \in \pi^{-1}_{D,S}(S) : i < r\} < \#\{i \in S : i \leq r\}$
(ii) $\#\{(i, c) \in \pi^{-1}_{\tilde{D},T}(T) : i < r\} < \#\{i \in T : i \leq r\}$

By Proposition 2.7, (i) is equivalent to $(r, c) \in \pi^{-1}_{D,S}(S)$. If indeed $(r, c) \in \pi^{-1}_{D,S}(S)$ holds, then our induction step is complete. Otherwise, $(r, c) \notin \pi^{-1}_{D,S}(S)$, so by definition, $(r, c) \in \tilde{D}$. Thus, applying Proposition 2.7 to $\tilde{D}$, $T \subseteq [n]$ and $(r, c) \in \tilde{D}$, (ii) is equivalent to $(r, c) \in \pi^{-1}_{\tilde{D},T}(T)$. Hence, (5) holds for all $i \leq r$. □

Corollary 2.15. *Let $D \subseteq [n]^2$ and $S \subseteq U \subseteq [n]$. Then $\pi^{-1}_{D,S}(S) \subseteq \pi^{-1}_{D,U}(U)$.*

Proof. Take $T = U \setminus S$ in Lemma 2.14. □

Finally, we are ready to prove Theorem 2.13.

Proof of Theorem 2.13. ($\Leftarrow$) Let $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$. Then $\alpha_1 + \cdots + \alpha_n = \#D$ by Proposition 2.12. Also, for each $S \subseteq [n]$,

$$\sum_{i \in S} \alpha_i = \sum_{i \in S} \#\tau_D^{-1}(\{i\}) = \#\tau_D^{-1}(S),$$

so τ_D exhausts α over S. Thus, $\alpha \in \mathcal{S}_D$ by Theorem 2.10.

($\Rightarrow$) We induct on the sum of the row indices of each box in D, i.e., $\sum_{(i,j) \in D} i$. The base case of an empty diagram is trivial, so we may assume $D \neq \emptyset$. Then since $\alpha \in \mathcal{S}_D$, (2) implies $\alpha_1 + \cdots + \alpha_n = \#D > 0$, so we can choose m maximal such that $\alpha_m > 0$.

Case 1: (D contains boxes below row m). Pick $(r, c) \in D$ below row m (so $r > m$).

Claim 2.16. *There exists $r_1 < r$ such that $(r_1, c) \notin D$.*

Proof of Claim 2.16. By Theorem 2.10, there exists $\tau_{D,[m]} \in \mathsf{FCITab}(D)$ such that

$$\#\tau_{D,[m]}^{-1}([m]) \geq \alpha_1 + \cdots + \alpha_m = \alpha_1 + \cdots + \alpha_n = \#D. \tag{7}$$

Thus, $\tau_{D,[m]}(D) \subseteq [m]$. Consequently, by column-injectivity of $\tau_{D,[m]}$, there can be at most m boxes in each column of D. Since $(r, c) \in D$ with $r > m$, there are more than m boxes in column c if $(r_1, c) \in D$ for all $r_1 < r$. Hence there must be some $r_1 < r$ for which $(r_1, c) \notin D$, as asserted. □

By Claim 2.16, we can choose $r_1 < r$ maximal such that $(r_1, c) \notin D$. Let

$$\tilde{D} = (D \smallsetminus \{(r, c)\}) \cup \{(r_1, c)\}.$$

Claim 2.17. $\alpha \in \mathcal{S}_{\tilde{D}}$.

Proof of Claim 2.17. Since $\alpha \in \mathcal{S}_D$, $(r, c) \in D$, and $(r_1, c) \notin D$, we have

$$\alpha_1 + \cdots + \alpha_n = \#D = \#\tilde{D}.$$

Let $S \subseteq [n]$ and $T = S \cap [m]$. Then define $\tau_{\tilde{D},S} \in \mathsf{Tab}(\tilde{D})$ by

$$\tau_{\tilde{D},S}(i, j) = \begin{cases} \pi_{D,T}(r, c) & \text{if } (i, j) = (r_1, c), \\ \pi_{D,T}(i, j) & \text{otherwise.} \end{cases}$$

If $\pi_{D,T}(r, c) = \circ$, then certainly $\tau_{\tilde{D},S} \in \mathsf{FCITab}(\tilde{D})$. Otherwise, let $s = \pi_{D,T}(r, c)$. Since $(r, c) \in D$ but $r \notin T$, (r, c) contributes a ")" to $\mathsf{word}_{c,S}(D)$. Thus, by (3), (s, c) contributes an "(", so in particular $(s, c) \notin D$. From our choice of r_1, we must therefore have $s \leq r_1$, so $\tau_{\tilde{D},S}$ is flagged. Hence, $\tau_{\tilde{D},S} \in \mathsf{FCITab}(\tilde{D})$.

By construction,

$$\#\tau_{\tilde{D},S}^{-1}(\{i\}) = \#\pi_{D,T}^{-1}(\{i\})$$

for each $i \in [n]$, so $\tau_{\tilde{D},S}$ exhausts α over T by Theorem 2.10 and in particular Remark 2.11. Since $\alpha_i = 0$ for all $i > m$, we can write

$$\sum_{i \in S} \alpha_i = \sum_{i \in T} \alpha_i \leq \#\tau_{\tilde{D},S}^{-1}(T) \leq \#\tau_{\tilde{D},S}^{-1}(S).$$

Therefore, $\tau_{\tilde{D},S} \in \mathsf{FCITab}(\tilde{D})$ exhausts α over S, so $\alpha \in \mathcal{S}_{\tilde{D}}$ by Theorem 2.10. □

Since $r_1 < r$,

$$\sum_{(i,j) \in \tilde{D}} i < \sum_{(i,j) \in D} i.$$

Thus, Claim 2.17 and induction yields $\tau_{\tilde{D}} \in \mathsf{PerfectTab}(\tilde{D}, \alpha)$. Define $\tau_D \in \mathsf{Tab}(D)$ by

$$\tau_D(i, j) = \begin{cases} \tau_{\tilde{D}}(r_1, c) & \text{if } (i, j) = (r, c), \\ \tau_{\tilde{D}}(i, j) & \text{otherwise.} \end{cases}$$

Then it is easy to check that $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$, so Case 1 is complete.

Case 2: (D does not contain boxes below row m). We say an inequality $\sum_{i \in S} \alpha_i \leq \theta_D(S)$ from (2) is *nontrivial* if

$$\sum_{i \in S} \alpha_i > 0 \qquad \text{and} \qquad \theta_D(S) < \#D. \tag{8}$$

Case 2a: (All nontrivial inequalities from (2) are strict). Thus if (8) holds, then

$$\sum_{i\in S}\alpha_i < \theta_D(S). \tag{9}$$

Claim 2.18. *There exists $c \in [n]$ such that $(m,c) \in D$.*

Proof of Claim 2.18. By Theorem 2.10, there exists some $\tau_{D,\{m\}} \in \mathsf{FClTab}(D)$ which exhausts α over $\{m\}$. Then

$$\#\tau^{-1}_{D,\{m\}}(\{m\}) \geq \alpha_m > 0,$$

so $\tau_{D,\{m\}}(r,c) = m$ for some $(r,c)\in D$. Since $\tau_{D,\{m\}}$ is flagged, we must have $r \geq m$. But by the assumption of Case 2, there are no boxes below row m, so $r = m$. □

Pick $c \in [n]$ as in Claim 2.18. Then let $\tilde{D} = D \smallsetminus \{(m,c)\}$ and $\tilde{\alpha} = (\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n) := (\alpha_1,\ldots,\alpha_{m-1},\alpha_m - 1, 0,\ldots,0)$.

Claim 2.19. $\tilde{\alpha} \in \mathcal{S}_{\tilde{D}}$.

Proof of Claim 2.19. Since $\alpha_i = 0$ for all $i > m$, and $(m,c)\in D$, we have

$$\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_n = \alpha_1 + \cdots + \alpha_n - 1 = \#D - 1 = \#\tilde{D}. \tag{10}$$

For each $S \subseteq [n]$, let

$$\tau_{\tilde{D},S} = \pi_{D,S}|_{\tilde{D}} \in \mathsf{FClTab}(\tilde{D})$$

be the restriction of $\pi_{D,S}$ to $\tilde{D}$. Then by Proposition 2.3,

$$\#\tau^{-1}_{\tilde{D},S}(S) \geq \#\pi^{-1}_{D,S}(S) - 1 = \theta_D(S) - 1. \tag{11}$$

If $\sum_{i\in S}\alpha_i = 0$, then

$$\sum_{i\in S}\tilde{\alpha}_i = 0 \leq \#\tau^{-1}_{\tilde{D},S}(S).$$

If $\theta_D(S) = \#D$, then by (10) and (11),

$$\sum_{i\in S}\tilde{\alpha}_i \leq \tilde{\alpha}_1 + \cdots + \tilde{\alpha}_n = \#D - 1 = \theta_D(S) - 1 \leq \#\tau^{-1}_{\tilde{D},S}(S).$$

Finally, if $\sum_{i\in S}\alpha_i > 0$ and $\theta_D(S) < \#D$, then (9) must hold, so by (9) and (11),

$$\sum_{i\in S}\tilde{\alpha}_i \leq \sum_{i\in S}\alpha_i \leq \theta_D(S) - 1 \leq \#\tau^{-1}_{\tilde{D},S}(S).$$

In all three cases, $\tau_{\tilde{D},S}$ exhausts $\tilde{\alpha}$ over S, so $\tilde{\alpha} \in \mathcal{S}_{\tilde{D}}$ by Theorem 2.10. □

By construction,

$$\sum_{(i,j)\in\tilde{D}} i < \sum_{(i,j)\in D} i.$$

Thus, Claim 2.19 and induction yield $\tau_{\tilde{D}} \in \mathsf{PerfectTab}(\tilde{D},\tilde{\alpha})$. Define $\tau_D \in \mathsf{Tab}(D)$ by

$$\tau_D(i,j) = \begin{cases} m & \text{if } (i,j) = (m,c), \\ \tilde{\tau}(i,j) & \text{otherwise.} \end{cases}$$

Clearly, τ_D is flagged, has content α, and satisfies $\tau_D^{-1}(\{\circ\}) = \emptyset$. The only potential obstruction to column-injectivity is that there could be some $r \neq m$ for which $\tau_D(r,c) = m$.

This is impossible, since τ_D is flagged, so such an r must be greater than m, but by the assumption of Case 2 there are no boxes below row m. Thus, $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$, so Case 2a is complete.

Case 2b: (There exists a tight, nontrivial inequality in (2)). Thus, there exists $A \subseteq [n]$ satisfying

$$0 < \sum_{i \in A} \alpha_i = \theta_D(A) < \#D. \tag{12}$$

Let $D^{(1)} = \pi_{D,A}^{-1}(A)$ and $D^{(2)} = D \setminus D^{(1)}$. Then for each $i \in [n]$, set

$$\alpha_i^{(1)} = \begin{cases} \alpha_i & \text{if } i \in A, \\ 0 & \text{if } i \notin A \end{cases} \qquad \text{and} \qquad \alpha_i^{(2)} = \begin{cases} \alpha_i & \text{if } i \notin A, \\ 0 & \text{if } i \in A. \end{cases}$$

Claim 2.20. $\alpha^{(1)} := (\alpha_1^{(1)}, \dots, \alpha_n^{(1)}) \in \mathcal{S}_{D^{(1)}}$.

Proof of Claim 2.20. By (12) and Proposition 2.3, we have

$$\alpha_1^{(1)} + \cdots + \alpha_n^{(1)} = \sum_{i \in A} \alpha_i = \theta_D(A) = \#\pi_{D,A}^{-1}(A) = \#D^{(1)}.$$

Let $S \subseteq [n]$ and $T = S \cap A$. Then set

$$\tau_{D^{(1)},S} = \pi_{D,T}|_{D^{(1)}} \in \mathsf{FCITab}(D^{(1)}).$$

By Corollary 2.15, $\pi_{D,T}^{-1}(T) \subseteq D^{(1)}$, so $\tau_{D^{(1)},S}^{-1}(T) = \pi_{D,T}^{-1}(T)$. Thus, by Remark 2.11, $\tau_{D^{(1)},S}$ exhausts α over T. Hence,

$$\sum_{i \in S} \alpha_i^{(1)} = \sum_{i \in T} \alpha_i \le \#\tau_{D^{(1)},S}^{-1}(T) \le \#\tau_{D^{(1)},S}^{-1}(S),$$

so $\tau_{D^{(1)},S}$ exhausts $\alpha^{(1)}$ over S, and consequently $\alpha^{(1)} \in \mathcal{S}_{D^{(1)}}$ by Theorem 2.10. □

Claim 2.21. $\alpha^{(2)} := (\alpha_1^{(2)}, \dots, \alpha_n^{(2)}) \in \mathcal{S}_{D^{(2)}}$.

Proof of Claim 2.21. By (12) and Proposition 2.3,

$$\alpha_1^{(2)} + \cdots + \alpha_n^{(2)} = \alpha_1 + \cdots + \alpha_n - \sum_{i \in A} \alpha_i = \#D - \theta_D(A) = \#D - \#\pi_{D,A}^{-1}(A) = \#D^{(2)}.$$

Let $S \subseteq [n]$, $T = S \setminus A$, and $U = A \cup T$. Then by Theorem 2.10, Remark 2.11, (12), Proposition 2.3, and Lemma 2.14, we can write

$$\begin{aligned} \sum_{i \in S} \alpha_i^{(2)} = \sum_{i \in U} \alpha_i - \sum_{i \in A} \alpha_i &\le \#\pi_{D,U}^{-1}(U) - \theta_D(A) \\ &= \#\pi_{D,U}^{-1}(U) - \#\pi_{D,A}^{-1}(A) = \#\pi_{D^{(2)},T}^{-1}(T) \le \#\pi_{D^{(2)},T}^{-1}(S). \end{aligned}$$

Thus, $\tau_{D^{(2)},S} := \pi_{D^{(2)},T}$ exhausts $\alpha^{(2)}$ over S, so $\alpha^{(2)} \in \mathcal{S}_{D^{(2)}}$ by Theorem 2.10. □

By (12) and Proposition 2.3, we have

$$0 < \#\pi_{D,A}^{-1}(A) < \#D,$$

so $D^{(1)}, D^{(2)} \subsetneq D$. Thus, by Claims 2.20 and 2.21 and induction, there exist

$$\tau_{D^{(1)}} \in \mathsf{PerfectTab}(D^{(1)}, \alpha^{(1)}) \text{ and } \tau_{D^{(2)}} \in \mathsf{PerfectTab}(D^{(2)}, \alpha^{(2)}).$$

Define $\tau_D = \tau_{D^{(1)}} \cup \tau_{D^{(2)}} \in \mathsf{Tab}(D)$ by

$$\tau_D(i,j) = \begin{cases} \tau_{D^{(1)}}(i,j) & \text{if } (i,j) \in D^{(1)}, \\ \tau_{D^{(2)}}(i,j) & \text{if } (i,j) \in D^{(2)}. \end{cases}$$

Clearly τ_D is flagged and satisfies $\tau_D^{-1}(\{\circ\}) = \emptyset$. It has content α because $\alpha = \alpha^{(1)} + \alpha^{(2)}$, and it is column-injective because the images of $\tau_{D^{(1)}}$ and $\tau_{D^{(2)}}$ are disjoint. Therefore, $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$ and Case 2b is complete.

This completes the proof of Theorem 2.13. □

2.2. **Polytopal descriptions of perfect tableaux.** Given $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, define the indicator polytope

$$\mathcal{P}(D, \alpha) \subseteq \mathbb{R}^{n^2}$$

to be the polytope with points of the form $(\alpha_{ij})_{i,j \in [n]} = (\alpha_{11}, \ldots, \alpha_{n1}, \ldots, \alpha_{1n}, \ldots, \alpha_{nn})$ governed by the inequalities (A)-(C) below.

(A) Column-Injectivity Conditions: For all $i, j \in [n]$,

$$0 \leq \alpha_{ij} \leq 1.$$

(B) Content Conditions: For all $i \in [n]$,

$$\sum_{j=1}^{n} \alpha_{ij} = \alpha_i.$$

(C) Flag Conditions: For all $s, j \in [n]$,

$$\sum_{i=1}^{s} \alpha_{ij} \geq \#\{(i,j) \in D : i \leq s\}.$$

Proposition 2.22. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ with $\alpha_1 + \cdots + \alpha_n = \#D$. If $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$, then for each $j \in [n]$, we have*

$$\sum_{i=1}^{n} \alpha_{ij} = \#\{(i,j) \in D : i \in [n]\}.$$

Proof. From the flag conditions (C) where $s = n$, we have that

$$\sum_{i=1}^{n} \alpha_{ij} \geq \#\{(i,j) \in D : i \in [n]\}.$$

If this inequality is strict for any j, then using the content conditions (B), we can write

$$\#D = \alpha_1 + \cdots + \alpha_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ij} > \sum_{j=1}^{n} \#\{(i,j) \in D : i \in [n]\} = \#D,$$

a contradiction. □

Theorem 2.23. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$.*

Proof. (⇒) By Proposition 2.12, we have $\alpha_1 + \cdots + \alpha_n = \#D$. Let $\tau \in \mathsf{PerfectTab}(D, \alpha)$. Then for each $i, j \in [n]$, set

$$\alpha_{ij} = \#\{r \in [n] : \tau(r, j) = i\} = \begin{cases} 1 & \text{if } \tau(r, j) = i \text{ for some } r \in [n], \\ 0 & \text{otherwise,} \end{cases}$$

where the second equality follows from the fact that τ is column-injective.

Claim 2.24. $(\alpha_{ij}) \in \mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2}$.

Proof of Claim 2.24. Clearly $(\alpha_{ij}) \in \mathbb{Z}^{n^2}$ and the column-injectivity conditions (A) hold. Since τ has content α,

$$\sum_{j=1}^{n} \alpha_{ij} = \sum_{j=1}^{n} \#\{r \in [n] : \tau(r, j) = i\} = \#\tau^{-1}(\{i\}) = \alpha_i$$

for each $i \in [n]$, so the content conditions (B) hold. Finally, for each $s, j \in [n]$, we have

$$\sum_{i=1}^{s} \alpha_{ij} = \#\{r \in [n] : \tau(r, j) \leq s\} \geq \#\{(r, j) \in D : r \leq s\}$$

since τ is flagged. Thus, the flag conditions (C) also hold. □

(⇐) Let $(\alpha_{ij}) \in \mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2}$. By the column-injectivity conditions (A), $\alpha_{ij} \in \{0, 1\}$. Thus, by Proposition 2.22, there exists for each $j \in [n]$ a bijection

$$\varphi_j : \{i \in [n] : (i, j) \in D\} \to \{i \in [n] : \alpha_{ij} = 1\}$$

that is order-preserving, i.e., φ_j satisfies $\varphi_j(i) < \varphi_j(i')$ whenever $i < i'$. Define $\tau \in \mathsf{Tab}(D)$ by $\tau(i, j) = \varphi_j(i)$.

Claim 2.25. $\tau \in \mathsf{PerfectTab}(D, \alpha)$.

Proof of Claim 2.25. By construction, $\tau^{-1}(\{\circ\}) = \emptyset$. Since φ_j is injective and order-preserving, τ is strictly increasing along columns, hence column-injective. For each $i \in [n]$, the content conditions (B) imply

$$\tau^{-1}(\{i\}) = \sum_{j=1}^{n} \#\varphi_j^{-1}(\{i\}) = \sum_{j=1}^{n} \alpha_{ij} = \alpha_i,$$

so τ has content α. Finally, the flag conditions (C) show that for each $s, j \in [n]$,

$$\#\{i \leq s : (i, j) \in D\} \leq \sum_{i=1}^{s} \alpha_{ij} = \#\{i \leq s : \alpha_{ij} = 1\},$$

so $\varphi_j(i) \leq i$ for each $(i, j) \in D$ since φ_j is order-preserving. Thus, $\tau(i, j) = \varphi_j(i) \leq i$ and τ is flagged. Hence, $\tau \in \mathsf{PerfectTab}(D, \alpha)$. □

This shows that $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ and completes the proof of the theorem. □

Remark 2.26. The proof of Claim 2.25 shows that if $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$, then we can find $\tau \in \mathsf{PerfectTab}(D, \alpha)$ which is not only column-injective, but also strictly increasing along columns, so $\tau(i, j) < \tau(i', j)$ whenever $i < i'$. Thus $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ if and only if $\mathsf{PerfectTab}(D, \alpha)_{\downarrow} \neq \emptyset$.

Theorem 2.23 formulates the problem of determining if $\mathsf{PerfectTab}(D,\alpha) \neq \emptyset$ in terms of feasibility of an integer linear programming problem. In general, integral feasibility is NP-complete. We now show that in our case, feasibility of the problem is equivalent to feasibility of its LP-relaxation:

Theorem 2.27. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha_1 + \cdots + \alpha_n = \#D$. Then $\mathcal{P}(D,\alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$ if and only if $\mathcal{P}(D,\alpha) \neq \emptyset$.*

For reasons given in the Introduction, we provide two proofs of this fact.

Proof 1 of Theorem 2.27. We write the constraints (A)-(C) in the form $M\vec{x} \leq \vec{b}$ where M is a $(3n^2+n) \times n^2$ block matrix and $\vec{b}$ is a vector of length $3n^2+n$ of the form

$$M = \begin{pmatrix} M_{A_1} \\ M_{A_2} \\ M_B \\ M_C \end{pmatrix} \text{ and } \vec{b} = (b_i)_{i=1}^{3n^2+n}.$$

Let $\vec{b}_I$ denote the subvector of $\vec{b}$ containing those b_i with $i \in I \subseteq [3n^2+n]$. Also, we use the following coordinatization:

$$\vec{x} = (\alpha_{11}, \ldots, \alpha_{n1}, \alpha_{12}, \ldots, \alpha_{n2}, \ldots, \alpha_{nn})^T.$$

- M_{A_1} is the $n^2 \times n^2$ block corresponding to the condition $0 \leq \alpha_{ij}$ from (A). Thus, $M_{A_1} = -I_{n^2}$ and $b_r = 0$ for $r \in [1, n^2]$.
- M_{A_2} is the $n^2 \times n^2$ block corresponding to $\alpha_{ij} \leq 1$ from (A). Hence, $M_{A_2} = I_{n^2}$ and $b_r = 1$ for $r \in [n^2+1, 2n^2]$.
- M_C is the $n^2 \times n^2$ matrix for (C). Thus,

$$M_C = \begin{pmatrix} M_{C_T} & 0 & \ldots & 0 \\ 0 & M_{C_T} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{C_T} \end{pmatrix}$$

  where $M_{C_T} = (c_{ij})_{1 \leq i,j \leq n}$ is lower triangular such that $c_{ij} = -1$ for $i \geq j$. Also,

$$b_{(2n^2+n)+n(j-1)+s} = -\#\{(i,j) \in D : i \leq s\}, \text{ for } s, j \in [n].$$

- M_B is the $n \times n^2$ block encoding (B). Take $M_B = \begin{pmatrix} I_n & I_n & \ldots & I_n \end{pmatrix}$ and $\vec{b}_{[2n^2+1, 2n^2+n]} = (\alpha_i)_{i \in [n]}$. Clearly $M_B\vec{x} \leq (\alpha_i)_{i \in [n]}$ encodes the inequalities $\sum_{j=1}^n \alpha_{ij} \leq \alpha_i$. Now, (B) requires $\sum_{j=1}^n \alpha_{ij} = \alpha_i$. However, $\alpha_1 + \cdots + \alpha_n = \#D$ ensures that

$$\begin{pmatrix} M_B \\ M_C \end{pmatrix} \vec{x} \leq \vec{b}_{[2n^2+1, 3n^2+n]} \text{ only if } M_B\vec{x} = (\alpha_i)_{i \in [n]}.$$

Summarizing, $M\vec{x} \leq \vec{b}$ indeed encodes (A)-(C).

Example 2.28. For $n = 2$ consider $\vec{x} = (\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22})^T$ with $D = \{(1,1), (1,2), (2,2)\} \subset [2] \times [2]$ and $\alpha = (2,1)$.

We have

$$M_{A_1}\vec{x} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$M_{A_2}\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M_B\vec{x} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \leq \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$M_C\vec{x} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \leq \begin{pmatrix} -\#\{(i,1) \in D : i \leq 1\} \\ -\#\{(i,1) \in D : i \leq 2\} \\ -\#\{(i,2) \in D : i \leq 1\} \\ -\#\{(i,2) \in D : i \leq 2\} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -2 \end{pmatrix}$$

Theorem 2.29. *M is a totally unimodular matrix; that is, every minor of M equals $0, 1$, or -1.*

Proof. Suppose M' is a square submatrix of M with k rows from M_{A_1} or M_{A_2}. We show by induction on k that $\det(M') \in \{0, \pm 1\}$.

For the base case $k = 0$, consider M' an $\ell \times \ell$ submatrix of M with only rows from M_B and M_C. Let M'_B, M'_C be the corresponding blocks of M', i.e. $M' = \begin{pmatrix} M'_B \\ M'_C \end{pmatrix}$ where M'_B, or M'_C, is the submatrix of M_B, or M_C respectively, using the rows and columns of M'. Since M_B has one 1 per column, M'_B has at most one 1 per column. By the form of M_C, it is straightforward to row reduce M'_C to obtain a $(0, -1)$-matrix M''_C with at most one -1 in each column. Let $M'' = \begin{pmatrix} M'_B \\ M''_C \end{pmatrix}$, an $\ell \times \ell$ matrix. It is textbook (see [11, Theorem 13.3]) that if a $(0, \pm 1)$-matrix N has at most one 1 and at most one -1 in each column, N is totally unimodular; hence $\det(M') = \pm \det(M'') \in \{0, -1, 1\}$ as desired. Thus the base case holds.

Now suppose M' is a square submatrix of M that contains $k \geq 1$ rows from M_{A_1} or M_{A_2}. Let R be such a row from M_{A_1} or M_{A_2}. If R contains all 0's, $\det(M') = 0$, and we are done. Otherwise R contains a single ± 1. Hence the cofactor expansion for $\det(M')$ along R gives $\det(M') = \pm \det(M'')$ where M'' is a submatrix of M with $k - 1$ rows from M_{A_1} or M_{A_2}. So by induction, $\det(M') \in \{0, \pm 1\}$, as required. □

Since M is totally unimodular then any vertices of $M\vec{x} \leq \vec{b}$ are integral [11, Theorem 13.2]. Thus, if $\mathcal{P}(D, \alpha) \neq \emptyset$ then its vertices are integral, i.e., $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$. □

Proof 2 of Theorem 2.27. Given a point $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$, we say a pair of sequences

$$(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m) \in [n]^{m+1} \times [n]^m,$$

for some $m \in \mathbb{Z}_{>0}$, is *stable* at (α_{ij}) if the properties (i)-(iv) below hold. The purpose of each property will become clear later.

(i) $r_{m+1} = r_1$.
(ii) For all $k \in [m]$, $\alpha_{r_k c_k}, \alpha_{r_{k+1} c_k} \notin \mathbb{Z}$.
(iii) For all $k \in [m]$, if $i > r_{k+1}$ and $\alpha_{i c_k} \notin \mathbb{Z}$, then $i = r_k$.

(iv) There exists $(r,c) \in [n]^2$ such that

$$\#\{k \in [m] : (r,c) = (r_k, c_k)\} \neq \#\{k \in [m] : (r,c) = (r_{k+1}, c_k)\}.$$

Claim 2.30. *For any* $(\alpha_{ij}) \in \mathcal{P}(D,\alpha) \smallsetminus \mathbb{Z}^{n^2}$*, there exists* $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ *stable at* (α_{ij})*.*

Proof of Claim 2.30. Choose r_1, c_1 such that $\alpha_{r_1c_1} \notin \mathbb{Z}$, and assume that we have fixed r_k, c_k such that $\alpha_{r_kc_k} \notin \mathbb{Z}$. By Proposition 2.22, we have

$$\sum_{i=1}^{n} \alpha_{ic_k} = \#\{(i, c_k) \in D : i \in [n]\} \in \mathbb{Z}.$$

Thus, as $\alpha_{r_kc_k} \notin \mathbb{Z}$, it makes sense to set

$$r_{k+1} = \max\{i \neq r_k : \alpha_{ic_k} \notin \mathbb{Z}\}. \tag{13}$$

If $r_{k+1} = r_\ell$ for some $\ell \in [k]$, then end the construction of these sequences. Otherwise, the content conditions (B) say that

$$\sum_{j=1}^{n} \alpha_{r_{k+1}j} = \alpha_{r_{k+1}} \in \mathbb{Z},$$

and since $\alpha_{r_{k+1}c_k} \notin \mathbb{Z}$, we can choose $c_{k+1} \neq c_k$ such that $\alpha_{r_{k+1}c_{k+1}} \notin \mathbb{Z}$, completing the recursive definition. By the pigeonhole principle, this process must halt, yielding sequences $r_1, \ldots, r_\ell, \ldots, r_{m+1}$ and $c_1, \ldots, c_\ell, \ldots, c_m$ with $r_{m+1} = r_\ell$.

By disregarding the first $\ell - 1$ terms of each sequence, we may assume $\ell = 1$ without loss of generality. Then we assert that $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ is stable at (α_{ij}). Indeed, (i) and (ii) are immediate from the construction, (iii) follows from (13), and (iv) holds because $(r,c) := (r_2, c_2)$ exists and satisfies

$$\#\{k \in [m] : (r,c) = (r_k, c_k)\} = 1 \qquad \text{and} \qquad \#\{k \in [m] : (r,c) = (r_{k+1}, c_k)\} = 0,$$

since $c_2 \neq c_1$ and $r_2 \neq r_k$ for all $k \neq 2$. □

We now fix a pair of sequences $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$. Given (α_{ij}) and $\delta > 0$, set

$$\alpha_{ij}^{\delta} = \alpha_{ij} + \delta[\#\{k \in [m] : (i,j) = (r_k, c_k)\} - \#\{k \in [m] : (i,j) = (r_{k+1}, c_k)\}]. \tag{14}$$

Claim 2.31. *If* $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ *is stable at* $(\alpha_{ij}) \in \mathcal{P}(D,\alpha)$*, then* $(\alpha_{ij}^{\delta}) \in \mathcal{P}(D,\alpha)$ *for some* $\delta > 0$*.*

Proof of Claim 2.31. First, note that the content conditions (B) are preserved regardless of our choice of δ. Indeed, for each $i \in [n]$,

$$\begin{aligned}\sum_{j=1}^{n} \alpha_{ij}^{\delta} &= \sum_{j=1}^{n} [\alpha_{ij} + \delta[\#\{k \in [m] : (i,j) = (r_k, c_k)\} - \#\{k \in [m] : (i,j) = (r_{k+1}, c_k)\}]] \\ &= \alpha_i + \delta[\#\{k \in [m] : i = r_k\} - \#\{k \in [m] : i = r_{k+1}\}],\end{aligned}$$

and the term in brackets vanishes by (i).

We next check the flag conditions (C). For each $s, j \in [n]$, we can write

$$\sum_{i=1}^{s} \alpha_{ij}^{\delta} = \sum_{i=1}^{s} [\alpha_{ij} + \delta[\#\{k \in [m] : (i,j) = (r_k, c_k)\} - \#\{k \in [m] : (i,j) = (r_{k+1}, c_k)\}]]$$

$$= \sum_{i=1}^{s} \alpha_{ij} + \delta[\#\{k \in [m] : s \geq r_k \text{ and } j = c_k\} - \#\{k \in [m] : s \geq r_{k+1} \text{ and } j = c_k\}]$$

$$\geq \sum_{i=1}^{s} \alpha_{ij} - \delta[\#\{k \in [m] : r_{k+1} \leq s < r_k \text{ and } j = c_k\}]. \tag{15}$$

Thus, if $\#\{k \in [m] : r_{k+1} \leq s < r_k \text{ and } j = c_k\} = 0$, then the flag condition (C) for these s, j is preserved.

Otherwise, $r_{k+1} \leq s < r_k$ and $j = c_k$ for some $k \in [m]$, so (ii) and (iii) tell us that there is exactly one $i > s$ for which $\alpha_{ij} \notin \mathbb{Z}$, namely $i = r_k$. This, combined with Proposition 2.22, shows that

$$\sum_{i=1}^{s} \alpha_{ij} = \sum_{i=1}^{n} \alpha_{ij} - \sum_{i=s+1}^{n} \alpha_{ij} = \#\{(i,j) \in D : i \in [n]\} - \sum_{i=s+1}^{n} \alpha_{ij} \notin \mathbb{Z}. \tag{16}$$

By the nonintegrality from (16), the flag inequalities (C) for $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$ are strict:

$$\sum_{i=1}^{s} \alpha_{ij} > \#\{(i,j) \in D : i \leq s\}. \tag{17}$$

Hence, by taking δ sufficiently small and applying (15) and (17), we can ensure

$$\sum_{i=1}^{s} \alpha_{ij}^{\delta} \geq \sum_{i=1}^{s} \alpha_{ij} - \delta[\#\{k \in [m] : r_{k+1} \leq s < r_k \text{ and } j = c_k\}] \geq \#\{(i,j) \in D : i \leq s\}$$

for all $s, j \in [n]$, so the flag conditions (C) will be preserved. If $\alpha_{ij} \neq \alpha_{ij}^{\delta}$ then by (14) we must have $(i,j) = (r_k, c_k)$ or $(i,j) = (r_{k+1}, c_k)$ for some k, which by (ii) implies $0 < \alpha_{ij} < 1$. So we can require in addition that δ be small enough that $0 \leq \alpha_{ij}^{\delta} \leq 1$ for all $i, j \in [n]$. For such δ, the conditions (A)-(C) all hold, so $(\alpha_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)$. □

Finally, choose a point $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$ with the maximum number of integer coordinates. If $(\alpha_{ij}) \in \mathbb{Z}^{n^2}$, then we are done. Otherwise, there exists $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ that is stable at (α_{ij}) by Claim 2.30. By (iv), there exists $(r, c) \in [n]^2$ such that $|\alpha_{rc}^{\delta}| \to \infty$ as $\delta \to \infty$, so α_{rc}^{δ} violates the column-injectivity conditions (A) for large δ. This, combined with Claim 2.31, shows that the set $S = \{\delta > 0 : (\alpha_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)\}$ is nonempty and bounded above. Thus, we can define $\eta = \sup S$ and set $(\tilde{\alpha}_{ij}) = (\alpha_{ij}^{\eta})$. Since $\mathcal{P}(D, \alpha)$ is closed and the map $\delta \mapsto (\alpha_{ij}^{\delta})$ from S to $\mathcal{P}(D, \alpha)$ is continuous, this supremum is in fact a maximum, and $(\tilde{\alpha}_{ij}) \in \mathcal{P}(D, \alpha)$. By our choice of (α_{ij}), we cannot have $\tilde{\alpha}_{r_k c_k} \in \mathbb{Z}$ or $\tilde{\alpha}_{r_{k+1} c_k} \in \mathbb{Z}$ for any $k \in [m]$, since then $(\tilde{\alpha}_{ij})$ has more integer coordinates than (α_{ij}). Thus, $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ is stable at $(\tilde{\alpha}_{ij})$, so by Claim 2.31, there exists $\delta > 0$ for which $(\tilde{\alpha}_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)$. But then $(\alpha_{ij}^{\eta+\delta}) = (\tilde{\alpha}_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)$, contradicting the maximality of η. □

In summary, applying the results of this section to $D = D(w)$,

$$c_{\alpha,w} > 0 \overset{[2]}{\Longleftrightarrow} \alpha \in \mathcal{S}_D \overset{\text{Thm } 2.13}{\Longleftrightarrow} \mathsf{PerfectTab}(D, \alpha) \neq \emptyset \overset{\text{Thm } 2.23}{\Longleftrightarrow} \mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset \overset{\text{Thm } 2.27}{\Longleftrightarrow} \mathcal{P}(D, \alpha) \neq \emptyset. \tag{18}$$

If $D \subseteq [n]^2$ has many identical columns, then many of the flag conditions (C) will look essentially the same. Thus, for efficiency of computation, we construct a "compressed" version of $\mathcal{P}(D, \alpha)$ that removes some of the repetitive inequalities.

A tuple $\mathcal{C} = (m, \{P_k\}_{k=1}^{\ell}, \{p_k\}_{k=1}^{\ell}, \{\lambda_k\}_{k=1}^{\ell})$ is a *compression* of $D \subseteq [n]^2$ if:

- $m \leq n$ is a nonnegative integer such that $(r, p) \notin D$ whenever $r > m$ and $p \in [n]$,
- $P = P_1 \dot\cup \cdots \dot\cup P_\ell \subseteq [n]$ such that if $p, p' \in P_k$ then
$$\{r \in [n] : (r, p) \in D\} = \{r \in [n] : (r, p') \in D\},$$
and moreover if D is nonempty in column p then $p \in P_k$ for some $k \in [\ell]$.
- $p_k \in P_k$ a representative for each $k \in [\ell]$, and
- $\lambda_k = \# P_k$ for each $k \in \ell$.

For $D \subseteq [n]^2$, a compression $\mathcal{C}$ of D, and $\tilde\alpha = (\tilde\alpha_1, \ldots, \tilde\alpha_m) \in \mathbb{Z}_{\geq 0}^m$ define

$$\mathcal{Q}(D, \mathcal{C}, \tilde\alpha) \subseteq \mathbb{R}^{m\ell} \tag{19}$$

to be the polytope with points of the form $(\tilde\alpha_{ik})_{i \in [m], k \in [\ell]}$ satisfying (A′)-(C′) below.

(A′) Column-Injectivity Conditions: For all $i \in [m], k \in [\ell]$,
$$0 \leq \tilde\alpha_{ik} \leq 1.$$

(B′) Content Conditions: For all $i \in [m]$,
$$\sum_{k=1}^{\ell} \lambda_k \tilde\alpha_{ik} = \alpha_i.$$

(C′) Flag Conditions: For all $s \in [m], k \in [\ell]$,
$$\sum_{i=1}^{s} \tilde\alpha_{ik} \geq \#\{(i, p_k) \in D : i \leq s\}.$$

Remark 2.32. We can always take $m = \ell = n$ and $P_k = \{k\}$ for each $k \in [\ell]$, in which case $\mathcal{Q}(D, \mathcal{C}, \tilde\alpha) = \mathcal{P}(D, \alpha) \subseteq \mathbb{R}^{n^2}$.

Theorem 2.33. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ and $\tilde\alpha = (\tilde\alpha_1, \ldots, \tilde\alpha_m) := (\alpha_1, \ldots, \alpha_m)$. Then $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathcal{P}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_m = \#D$, $\alpha_{m+1} = \cdots = \alpha_n = 0$, and $\mathcal{Q}(D, \mathcal{C}, \tilde\alpha) \neq \emptyset$.*

Proof. ($\Rightarrow$) Let $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$. Then by the content and flag conditions (B) and (C),

$$\begin{aligned}\#D = \alpha_1 + \cdots + \alpha_n \geq \alpha_1 + \cdots + \alpha_m &= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_{ij} \geq \sum_{j=1}^{n} \#\{(i, j) \in D : i \leq m\} = \#D.\end{aligned}$$

Thus, $\alpha_1 + \cdots + \alpha_m = \#D$ and $\alpha_{m+1} = \cdots = \alpha_n = 0$. Now, for each $i \in [m]$ and $k \in [\ell]$, set

$$\tilde\alpha_{ik} = \frac{1}{\lambda_k} \sum_{j \in P_k} \alpha_{ij}.$$

We claim that $(\tilde{\alpha}_{ik}) \in \mathcal{Q}(D, \mathcal{C}, \alpha)$. First, for each $i \in [m]$ and $k \in [\ell]$, we have

$$0 \le \tilde{\alpha}_{ik} = \frac{1}{\lambda_k} \sum_{j \in P_k} \alpha_{ij} \le \frac{1}{\lambda_k} \sum_{j \in P_k} 1 = 1,$$

so the column-injectivity conditions (A′) are satisfied. Next, for each $i \in [m]$, (B) implies

$$\sum_{k=1}^{\ell} \lambda_k \tilde{\alpha}_{ik} = \sum_{k=1}^{\ell} \sum_{j \in P_k} \alpha_{ij} = \sum_{j=1}^{n} \alpha_{ij} = \alpha_i,$$

so the content conditions (B′) are satisfied. Finally, for each $s \in [m]$ and $k \in [\ell]$, (C) implies

$$\sum_{i=1}^{s} \tilde{\alpha}_{ik} = \frac{1}{\lambda_k} \sum_{j \in P_k} \sum_{i=1}^{s} \alpha_{ij} \ge \frac{1}{\lambda_k} \sum_{j \in P_k} \#\{(i,j) \in D : i \le s\} = \#\{(i, p_k) \in D : i \le s\},$$

so the flag conditions (C′) are satisfied.

($\Leftarrow$) Clearly $\alpha_1 + \cdots + \alpha_n = \#D$. Let $(\tilde{\alpha}_{ik}) \in \mathcal{Q}(D, \mathcal{C}, \tilde{\alpha})$. For each $i, j \in [n]$, set

$$\alpha_{ij} = \begin{cases} 0 & \text{if } i > m, \\ \tilde{\alpha}_{ik} & \text{if } i \le m \text{ and } j \in P_k. \end{cases}$$

We claim that $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$. The column-injectivity conditions (A) are clear. If $i > m$,

$$\sum_{j=1}^{n} \alpha_{ij} = 0 = \alpha_i.$$

Otherwise $i \le m$, and (B′) implies

$$\sum_{j=1}^{n} \alpha_{ij} = \sum_{k=1}^{\ell} \sum_{j \in P_k} \tilde{\alpha}_{ik} = \sum_{k=1}^{\ell} \lambda_k \tilde{\alpha}_{ik} = \alpha_i.$$

Thus, the content conditions (B) hold. Finally, if $s \in [n]$ and $j \in P_k$, then (C′) implies

$$\sum_{i=1}^{s} \alpha_{ij} = \sum_{i=1}^{\min\{s,m\}} \tilde{\alpha}_{ik} \ge \#\{(i, p_k) \in D : i \le \min\{s, m\}\} = \#\{(i,j) \in D : i \le s\}.$$

Hence, the flag conditions (C) hold as well. □

2.3. **Deciding membership in the Schubitope.** We use the above results of this section to give a polynomial time algorithm to check if a lattice point is in the Schubitope.

Let $D \subseteq [n]^2$, and fix a compression $\mathcal{C} = (m, \{P_k\}_{k=1}^{\ell}, \{p_k\}_{k=1}^{\ell}, \{\lambda_k\}_{k=1}^{\ell})$ of D (as in Section 2.2).

Theorem 2.34. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\ge 0}^{n}$. Then $\alpha \in \mathcal{S}_D$ if and only if $\alpha_1 + \cdots + \alpha_m = \#D$, $\alpha_{m+1} = \cdots = \alpha_n = 0$, and $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha}) \ne \emptyset$, where $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) := (\alpha_1, \ldots, \alpha_m)$.*

Proof. This follows from Theorems 2.13, 2.23, 2.27, and 2.33. □

For each $k \in [\ell]$, let $R_k(\mathcal{C}) = \{r \in [n] : (r, p_k) \in D\} \subseteq [m]$.

Theorem 2.35. *Given as input $\{R_k(\mathcal{C})\}_{k=1}^{\ell}$, $\{\lambda_k\}_{k=1}^{\ell}$, and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{Z}_{\ge 0}^{m}$ satisfying $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_m = \#D$, one can decide if $\alpha := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, 0, \ldots, 0) \in \mathbb{Z}_{\ge 0}^{n}$ lies in $\mathcal{S}_D$ in polynomial time in m and ℓ.*

Remark 2.36. In view of Theorem 2.34, this input is most natural, because the conditions $\alpha_1 + \cdots + \alpha_m = \#D$ and $\alpha_{m+1} = \cdots = \alpha_n = 0$ are clearly necessary, and it contains the minimum amount of information we need to compute $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha})$.

Remark 2.37. As in Remark 2.32, we can take $m = \ell = n$ and $P_k = \{k\}$ for each $k \in [\ell]$, so we can check if α is in $\mathcal{S}_D$ in polynomial time in n regardless of the structure of D.

Proof of Theorem 2.35. Since $R_k(\mathcal{C})$ takes m bits to encode for each $k \in [\ell]$, and $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha}) \subseteq \mathbb{R}^{m\ell}$ is governed by $O(m\ell)$ constraints, $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha})$ can be constructed in polynomial time in m and ℓ. By Theorem 2.34, we are done using L. Khachiyan's ellipsoid method [12]. □

3. Computing Rothe diagrams

We will repeatedly use the following to establish the complexity of computing preliminary data of $D(w)$ given $\mathsf{code}(w)$.

Proposition 3.1. *There exists an $O(L^2)$-time algorithm to compute $(w(1), \ldots, w(L))$ from the input $\mathsf{code}(w) = (c_1, \ldots, c_L)$.*

Proof. Clearly $w(1) = c_1 + 1$. After determining $w(1), \ldots, w(i-1)$, we determine (in $O(L)$-time) $\pi := \pi^{(i)} \in S_{i-1}$ such that $(w(\pi(1)) < w(\pi(2)) < \ldots < w(\pi(i-1)))$. Next, set

$$B := (w(\pi(1)), w(\pi(2)) - w(\pi(1)), w(\pi(3)) - w(\pi(2)), \ldots, w(\pi(i-1)) - w(\pi(i-2)))).$$

Let

$$V_t := \sum_{j=1}^{t} (B_j - 1), \quad \text{for } 0 \leq t \leq i-1.$$

Set $w(i) := c_i + T + 1$ where $T := \max_{t \in [0, i-1]}\{t \ : \ c_i \geq V_t\}$. By construction, $w(1), \ldots, w(i)$ is a partial permutation with code $(c_1, \ldots, c_{i-1}, c_i)$. Each stage $1 \leq i \leq L$ takes $O(i)$-time. □

The *essential set* of w consists of the maximally southeast boxes of each connected component of $D(w)$, i.e.,

$$\mathsf{Ess}(w) = \{(i,j) \in D(w) : (i+1, j), (i, j+1) \notin D(w)\}. \tag{20}$$

If it exists, we call the connected component of $D(w)$ involving $(1,1)$ the *dominant component* and denote it by $\mathsf{Dom}(w)$. For instance, in Example 1.1, $\mathsf{Dom}(w)$ has shape $(4,2,2,2)$. Further, if it exists, the *accessible box* $\mathbf{z}_w$ is the southmost then eastmost box in $\mathsf{Ess}(w) \smallsetminus \mathsf{Dom}(w)$. In Example 1.1,

$$\mathsf{Ess}(w) = \{(1,4), (3,4), (3,7), (4,2)\} \text{ and } \mathbf{z}_w = (3,7).$$

(Although $(4,2)$ is the southmost box of $\mathsf{Ess}(w)$, it is in $\mathsf{Dom}(w)$, and hence not the accessible.)

We will need the following in Section 5:

Proposition 3.2. *Given $\mathsf{code}(w)$, there exists an $O(L^2)$-time algorithm to compute $\mathbf{z}_w = (r, c)$ or determine it does not exist.*

Proof. Use Proposition 3.1 to find $(w(1), \ldots, w(L))$ in $O(L^2)$-time. Next, compute

$$w_{NW}(i) := \{w(j) : w(j) \leq w(i), \ j \leq i\}.$$

Then take

$$Y(i) := \{q - 1 \ : \ q \in w_{NW}(i)\} \smallsetminus w_{NW}(i), \quad \text{for } i \in [L].$$

Compute $k_i := \max Y(i)$ for $i \in [L]$ in $O(L^2)$-time (if $k_i \geq 1$, then k_i is the column index of the eastmost box of $D(w)$ in row i). In $O(L^2)$-time, calculate

$$I := \{i \in [2, \ldots, L] \ : \ k_i > \min_{j<i} w(j)\}.$$

Let $Y := \{(i, k_i) \ : \ i \in I\}$. Hence, $Y \cap \mathsf{Dom}(w) = \emptyset$. Thus, if $Y = \emptyset$, $\mathbf{z}_w$ does not exist. Otherwise, $\mathbf{z}_w \in Y$. Thus, in $O(L)$-time, determine $r := \max\{i : (i, k_i) \in Y\}$. Output $\mathbf{z}_w = (r, k_r)$. ☐

The *pivots* of $\mathbf{z}_w$ denoted $\mathsf{Piv}(\mathbf{z}_w)$ are the $\bullet$'s of $D(w)$ that are maximally southeast, among those northwest of $\mathbf{z}_w$. In Example 1.1, $\mathsf{Piv}((3,7)) = \{(2,3), (1,5)\}$.

4. PROOFS OF THEOREMS 1.2 AND 1.3

4.1. **Proof of Theorem 1.2.** By (1) combined with Theorem 2.35, it remains to establish the complexity of computing a compression of $D(w)$. For this, we need the following lemmas and propositions. Fix $w \in S_\infty$ with $\mathsf{code}(w) = (c_1, \ldots, c_L)$. Let $\sigma \in S_L$ be such that $\{w(\sigma(1)) < w(\sigma(2)) < \ldots < w(\sigma(L))\}$. For convenience, set $w(\sigma(0)) := 0$.

Lemma 4.1. *For $1 \leq h \leq L$, and for all*

$$j_1, j_2 \in \{w(\sigma(h-1)) + 1, w(\sigma(h-1)) + 2, \ldots, w(\sigma(h)) - 1\},$$

we have $(i, j_1) \in D(w)$ if and only if $(i, j_2) \in D(w)$.

Proof. For each k, let $u_1^{(k)} < \ldots < u_k^{(k)}$ be $w(1), w(2), \ldots, w(k)$ sorted in increasing order. Set $u_0^{(k)} := 0$. The lemma follows from the inductive claim that in the first k rows of $D(w)$, the columns $u_{h-1}^{(k)} + 1, u_{h-1}^{(k)} + 2, \ldots, u_h^{(k)} - 1$ are the same. The base case $k = 1$ is clear. The inductive step is straightforward by considering how, in row $k+1$ of $D(w)$, the $\bullet$ and its ray emanating east affects the columns. ☐

Define a collection of intervals in $[n]$ by

$$P'_{2k-1} := [w(\sigma(k-1)) + 1, w(\sigma(k)) - 1] \ \text{and} \ P'_{2k} := \{w(\sigma(k))\}, \ \text{for } 1 \leq k \leq L.$$

Let $1 \leq h_1 < h_2 < \ldots < h_\ell \leq 2L$ be indices of the intervals P'_h that are nonempty. Set $P_i := P'_{h_i}$.

Lemma 4.2. *If $j_1, j_2 \in P_k$ for some k, then $(i, j_1) \in D(w) \iff (i, j_2) \in D(w)$.*

Proof. This follows by the definition of $\{P_k\}_{k=1}^{\ell}$ together with Lemma 4.1. ☐

Let $p_k := \min\{p \in P_k\}$ for each $k \in [\ell]$.

Proposition 4.3. *There exists an $O(L^2)$-time algorithm to compute $\{P_k\}_{k=1}^{\ell}$, $\{p_k\}_{k=1}^{\ell}$, and $\{\#P_k\}_{k=1}^{\ell}$ from the input $\mathsf{code}(w) = (c_1, \ldots, c_L)$.*

Proof. Proposition 3.1 computes $(w(1), \ldots, w(L))$ in $O(L^2)$-time. It takes $O(L \log(L))$-time to sort $(w(1), \ldots, w(L))$, i.e., to compute $\sigma \in S_L$. Computing the endpoints, and thus cardinalities, of the P'_k takes $O(L)$-time as there are at most $2L$ of them. Then we reindex $\{\#P'_k\}_{k=1}^{2L}$ to obtain $\{\#P_k\}_{k=1}^{\ell}$ in $O(L)$-time. ☐

For each $k \in [\ell]$, let

$$R_k := \{r \in [L] : (r, p_k) \in D(w)\}.$$

Proposition 4.4. *Computing $\{R_k\}_{k=1}^{\ell}$ from $\mathsf{code}(w)$ takes $O(L^2)$-time.*

Proof. By $D(w)$'s definition, $r \in R_k$ if and only if $w(r) > p_k$ and $p_k \notin \{w(i) : i < r\}$. Propositions 4.3 and 3.1 give $\{P_k\}_{k=1}^{\ell}$, $\{p_k\}_{k=1}^{\ell}$ and $\{w(1), \ldots, w(L)\}$ in $O(L^2)$-time. □

Conclusion of proof of Theorem 1.2: Proposition 4.3 computes $\{P_k\}_{k=1}^{\ell}, \{p_k\}_{k=1}^{\ell}$, and $\{\#P_k\}_{k=1}^{\ell}$ in $O(L^2)$-time. Proposition 4.4 finds $\{R_k\}_{k=1}^{\ell}$ in $O(L^2)$-time. One checks, using Lemma 4.2, that $\mathcal{C} = (L, \{P_k\}_{k=1}^{\ell}, \{p_k\}_{k=1}^{\ell}, \{\#P_k\}_{k=1}^{\ell})$ is a compression of $D(w)$. Hence we may apply Theorem 2.35 by taking $D := D(w)$, $R_k(\mathcal{C}) := R_k$, $\lambda_k := \#P_k$ for $k \in [\ell]$ and $m := L$. Thus the result follows by (1). □

4.2. **Proof of Theorem 1.3; an application.** Remark 2.26 combined with (18) proves the theorem. □

Let $n_{132}(w)$ be the number of 132-patterns in $w \in S_n$, that is,

$$n_{132}(w) = \#\{(i,j,k) : 1 \leq i < j < k \leq n, w(i) < w(k) < w(j)\}.$$

Corollary 4.5. *There are at least $n_{132}(w) + 1$ distinct vectors α such that $c_{\alpha,w} > 0$.*

Proof. Suppose $i < j < k$ index a 132 pattern in w. There is a box b of $D(w)$ in row j and column $w(k)$. There are $N := n_{132}(w)$ many such boxes, $\mathsf{b}_1, \ldots, \mathsf{b}_N$ (all distinct), listed in English language reading order. Let M_i be boxes in the same column and connected component as b_i that are weakly north of b_i and strictly south of any b_j, where $j < i$. Iteratively define fillings $F_0, F_1, F_2, \ldots, F_N$ of $D(w)$:

(F_0) Fill each box c of $D(w)$ with the row number of c.
(F_i) For $1 \leq i \leq N$, F_i is the same as F_{i-1} except that $F_i(\mathsf{c}) := F_{i-1}(\mathsf{c}) - 1$ if $\mathsf{c} \in M_i$.

Clearly, $F_0 \in \mathsf{PerfectTab}_{\downarrow}(D(w)) := \bigcup_{\alpha} \mathsf{PerfectTab}_{\downarrow}(D(w), \alpha)$. Inductively assume $F_{i-1} \in \mathsf{PerfectTab}_{\downarrow}(D(w))$. Since labels only decrease, F_i satisfies the row bound condition. Next we check that each column is strictly increasing. Let m_i be the northmost box of M_i. If m_i is adjacent and directly below some b_j (for a $j < i$) then

$$F_i(\mathsf{b}_j) = F_0(\mathsf{b}_j) - 1 < F_0(\mathsf{m}_i) - 1 = F_i(\mathsf{m}_i),$$

as needed. Otherwise suppose m_i is adjacent and south of a non-diagram position. Let d_i (if it exists) be the first diagram box directly north of m_i. Then $F_0(\mathsf{d}_i) < F_0(\mathsf{m}_i) - 1$. Hence

$$F_i(\mathsf{d}_i) \leq F_0(\mathsf{d}_i) < F_0(\mathsf{m}_i) - 1 = F_i(\mathsf{m}_i),$$

verifying column increasingness here as well. That F_i is column increasing elsewhere is clear since F_{i-1} is column increasing (by induction) and only labels of M_i are changed.

It remains to check that every label of F_i is in $\mathbb{Z}_{>0}$. Since each box of $D(w)$ is decremented at most once, the only concern is there is a box x in the first row that appears in some M_i, since then $F_0(\mathsf{x}) = 1$ and $F_i(\mathsf{x}) = 0$. However, in this case b_i must be in $\mathsf{Dom}(w)$, which implies that the "1" in the 132-pattern associated to b_i could not exist, a contradiction. Thus $F_i \in \mathsf{PerfectTab}_{\downarrow}(D(w))$, completing the induction.

Finally, under Theorem 1.3, each F_i corresponds to a distinct exponent vector since the sum of the labels is strictly decreasing at each step $F_{i-1} \mapsto F_i$. □

From Corollary 4.5, this result of A. Weigandt [15] is immediate:

Corollary 4.6 (A. Weigandt's 132-bound). $\mathfrak{S}_w(1,1,1,\ldots,1) \geq n_{132}(w) + 1$.

As shown in [15], Corollary 4.6 in turn implies $\mathfrak{S}_w(1,1,\ldots,1) \geq 3$ if $n_{132}(w) \geq 2$, a recent conjecture of R. P. Stanley [13].

5. COUNTING $c_{\alpha,w}$ IS IN #P

5.1. **Vexillary permutations.** A permutation $w \in S_n$ is *vexillary* if there does not exist a 2143 *pattern*, i.e., indices $i < j < k < l$ such that w has the pattern $w(j) < w(i) < w(l) < w(k)$. For example, $w = \underline{538}412\underline{6}7$ is not vexillary; we underlined the positions of a 2143 pattern. *Fulton's criterion* states that w is vexillary if and only if there do not exist $(a,b),(c,d) \in \mathsf{Ess}(w)$ such that $a < c$ and $b < d$. In Example 1.1, w is not vexillary due to $(1,4)$ and $(3,7)$. Our main reference for this subsection is [8, Chapter 2].

We will also use this characterization of vexillary permutations:

Theorem 5.1. [6] *Given* $\mathsf{code}(w) = (c_1,\ldots,c_L) \in \mathbb{Z}^n_{\geq 0}$, *$w$ vexillary if and only if*

(i) *if i is such that $c_i > c_{i+1}$, then $c_i > c_j$ for any $j > i$, and*
(ii) *if i,h are such that $c_i \geq c_h$, then $\#\{j \ : \ i < j < h,\ c_j < c_h\} \leq c_i - c_h$.*

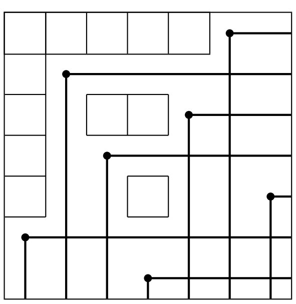
The *shape* of a vexillary permutation v is the partition $\lambda(v)$ formed by sorting $\mathsf{code}(v) = (c_1, c_2, \ldots)$ into decreasing order. Now, if $c_i \neq 0$, let e_i be the greatest integer $j \geq i$ such that $c_j \geq c_i$. The *flag*

$$\phi(v) = (\phi_1 \leq \phi_2 \leq \ldots \leq \phi_m)$$

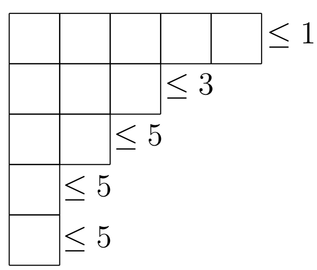
for v is the sequence of e_i's sorted into increasing order; see, e.g., [8, Definition 2.2.9].

Example 5.2. Consider $\mathsf{code}(v) = (5,1,3,1,2)$ for the vexillary $v = 6253714$. Here

$$e = (1,5,3,5,5), \phi(v) = (1,3,5,5,5) \text{ and } \lambda(v) = (5,3,2,1,1).$$



$D(v)$



$\lambda(v)$ flagged by $\phi(v)$

1	1	1	1	1
2	2	3		
3	4			
4				
5				

$T \in \mathsf{SSYT}(\lambda(v), \phi(v))$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0)$ and a flag $\phi = (\phi_1 \leq \phi_2 \leq \ldots \leq \phi_m)$ of positive integers, define the *flagged Schur function*

$$S_\lambda(\phi) = \det |h_{\lambda_i - i + j}(\phi_i)|_{i,j=1,\ldots,m},$$

where

$$h_k(n) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

is the complete homogeneous symmetric polynomial of degree k. Furthermore,

$$\mathfrak{S}_v = S_{\lambda(v)}(\phi(v)), \text{ for } v \text{ vexillary.} \tag{21}$$

A semistandard Young tableau of shape λ is *flagged* by ϕ if its entries in row i are $\leq \phi_i$; see Example 5.2. Denote the set of such tableaux by $\mathsf{SSYT}(\lambda, \phi)$. Then

$$S_\lambda(\phi) = \sum_{T \in \mathsf{SSYT}(\lambda,\phi)} x^{\mathsf{content}(T)}. \tag{22}$$

where $\mathsf{content}(T) = (\mu_1, \ldots, \mu_{\ell(\lambda)})$ such that μ_i is the number of i's in T.

5.2. **Graphical transition.** The transition recurrence for $\mathfrak{S}_w$ was found by A. Lascoux and M.-P. Schützenberger [6]. This is transition for the case discussed in [5]:

Theorem 5.3 ([6], cf. [5]). *Let $\mathbf{z}_w = (r, c)$ and $w' = w \cdot (r\ k)$ where $k = w^{-1}(c)$. Then*

$$\mathfrak{S}_w = x_r \mathfrak{S}_{w'} + \sum_{w'' = w' \cdot (i\ k)} \mathfrak{S}_{w''}, \tag{23}$$

where the summation is over $\{i : (i, w(i)) \in \mathsf{Piv}(\mathbf{z}_w)\}$.

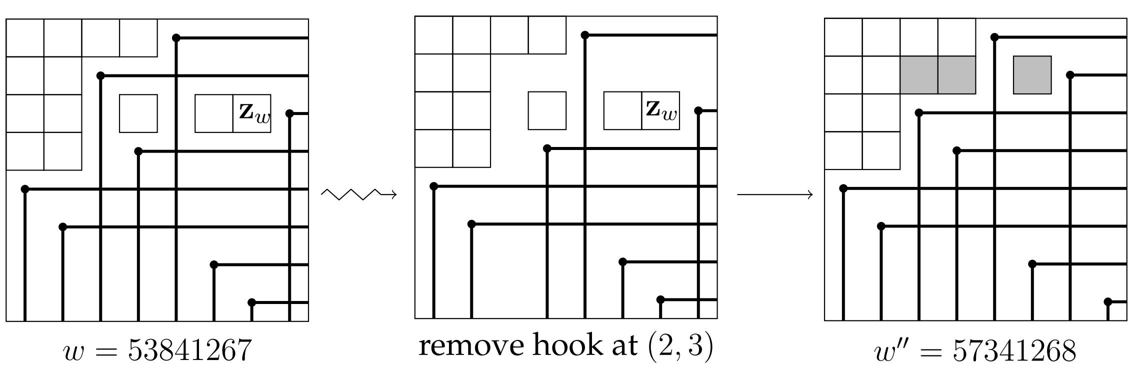
We will use the *graphical* transition tree $\mathcal{T}(w)$ of [5]. This reformulates (23) in terms of Rothe diagrams and certain moves on these diagrams. By definition, $D(w)$ (equivalently w) will label the root of $\mathcal{T}(w)$. If w is vexillary, stop. Otherwise, there exists an accessible box $\mathbf{z}_w = (r, c) \in D(w)$ (if not, $D(w) = \mathsf{Dom}(w)$, contradicting w is not vexillary).

The children of $D(w)$ are Rothe diagrams resulting from two types of moves:

- (T.1) *Deletion moves*: remove $\mathbf{z}_w$ from $D(w)$. The resulting diagram is $D(w')$. Add an edge $D(w) \xrightarrow{x_r} D(w')$.
- (T.2) *March moves*: There is a move for each $\mathbf{x}^{(i)} = (i, w(i)) \in \mathsf{Piv}(\mathbf{z}_w)$. Let $\mathcal{R}$ be the rectangle with corners $\mathbf{z}_w$ and $\mathbf{x}^{(i)}$. Remove $\mathbf{x}^{(i)}$ and its rays from $G(w)$ to form $G^{(i)}(w)$. Order the boxes $\{\mathsf{b}_i\}_{i=1}^r$ in $\mathcal{R}$ in English reading order. Move b_1 strictly north and strictly west to the closest position not occupied by other boxes of $D(w)$ or rays from $G^{(i)}(w)$. Repeat with $\mathsf{b}_2, \mathsf{b}_3, \ldots$ where b_j may move into a square left unoccupied by earlier moves. The resulting diagram will be $D(w'')$ where $w'' = w' \cdot (i\ k)$. Add an edge $D(w) \xrightarrow{i} D(w'')$.

Repeat for each child $D(u)$. Stop when u vexillary; these permutations are the leaves $\mathcal{L}(w)$ of $\mathcal{T}(w)$. (Multiple leaves may be labelled by the same permutation.)

Example 5.4. Let $w = 53841267$. We compute the march move 2 for the pivot $(2, 3)$:



The moved boxes during $D(w) \mapsto D(w'')$ are shaded gray.

Example 5.5. Let $w = 53861247$. Using $\mathcal{T}(w)$ from Figure 1, we compute

$$\begin{aligned}\mathfrak{S}_w = x_4 \cdot \mathfrak{S}_{73541268} + x_4 \cdot \mathfrak{S}_{57341268} + x_3^2 x_4 \cdot \mathfrak{S}_{53641278} + x_3 x_4 \cdot \mathfrak{S}_{63541278} + x_3 x_4 \cdot \mathfrak{S}_{56341278} \\ + \mathfrak{S}_{74531268} + \mathfrak{S}_{57431268} + x_3^2 \cdot \mathfrak{S}_{54631278} + x_3 \cdot \mathfrak{S}_{64531278} + x_3 \cdot \mathfrak{S}_{56431278}.\end{aligned}$$

For instance, $c_{(4,2,5,3),w} := [x_1^4 x_2^2 x_3^5 x_4^3]\mathfrak{S}_w = 1$ is witnessed by

- the path $w \xrightarrow{x_4} \bullet \xrightarrow{x_3} \bullet \xrightarrow{x_3} u = 53641278$, and
- the semistandard tableau

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2}\end{array}$$ of shape $\lambda(u)$, flagged by $\phi(u) = (1, 3, 4, 4)$.

Proposition 5.9 below formalizes a rule for $c_{\alpha,w}$ in terms of such pairs.

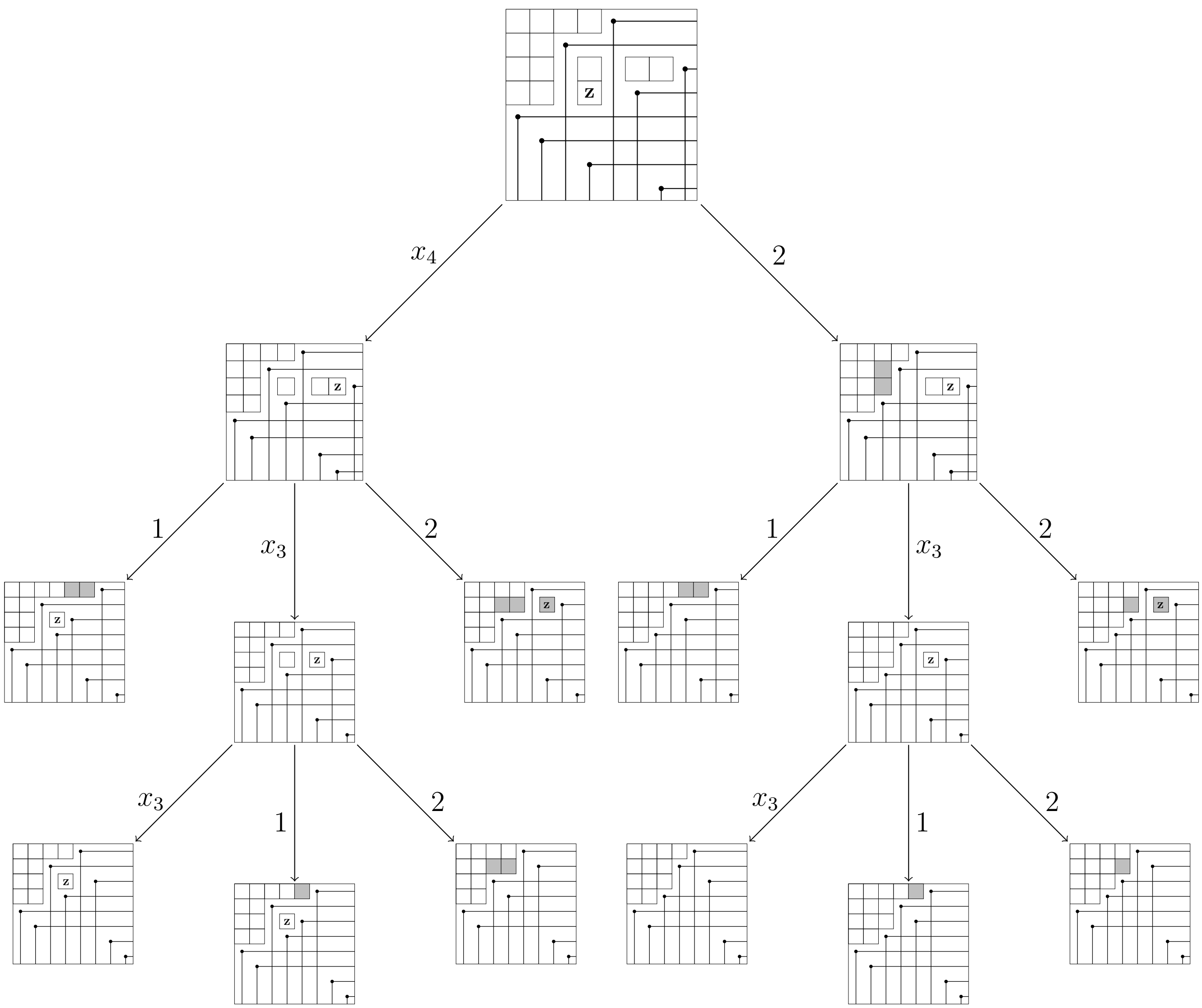


FIGURE 1. $\mathcal{T}(w)$ for $w = 53861247$ where the accessible boxes are marked with z and those boxes of the parent which moved are shaded gray.

5.3. **Proof of $\#\mathsf{P}$-ness.** The technical core of our proof of Theorem 1.5 is to show:

Theorem 5.6. *The problem of computing $c_{\alpha,w}$, given input α and* $\mathsf{code}(w)$, *is in* $\#\mathsf{P}$.

Define X to be the set consisting of pairs (S, R) where:

(X.1) $S = (s_1, \ldots, s_h)$, $s_t \in [L] \cup \{(x_k, m_t) \ : \ k \in [L], m_t \in \mathbb{Z}_{>0}\}$ such that if $s_t = (x_k, m_t)$ then $s_{t+1} \neq (x_k, m_{t+1})$ for $t < h$, and
(X.2) $R = (r_{ij})_{1\leq i,j\leq L}$, where $r_{ij} \in \mathbb{Z}_{\geq 0}$.

Fix $w \in S_\infty$ and a vexillary permutation $v \in S_\infty$. A (w, v)-*transition string* is a sequence $S = (s_1, \ldots, s_h)$ satisfying (X.1) such that if we interpret i as $\bullet \xrightarrow{i} \bullet$ and (x_k, m_t) as $\bullet \xrightarrow{x_k} \bullet \cdots \bullet \xrightarrow{x_k} \bullet$ (m_t-times) then S describes a path from w to (a leaf labelled by) v in $\mathcal{T}(w)$. Let $\mathsf{Trans}(w, v)$ be the set of such sequences.

The *deletion weight* of $S \in \mathsf{Trans}(w, v)$ is

$$\mathsf{delwt}(S) = \sum m_t \cdot \vec{e_r},$$

where the summation is over $1 \leq t \leq h$ such that $s_t = (x_r, m_t) \in S$ for some $r \in [L]$ (depending on t). Here $\vec{e_r} \in \mathbb{Z}^L_{\geq 0}$ is the r-th standard basis vector and L is the length of $\mathsf{code}(w) = (c_1, c_2, \ldots, c_L)$.

Example 5.7. In Figure 1 we read the $(w = 53861247, v = 54631278)$-transition string $S = (2, (x_3, 2))$ as the path $w \xrightarrow{2} \bullet \xrightarrow{x_3} \bullet \xrightarrow{x_3} v$. Here, $\mathsf{delwt}(S) = (0, 0, 2, 0)$.

Suppose T is a tableau of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_L \geq 0)$, with entries in $[L]$ and weakly increasing along rows. Define

$$R(T) = (r_{ij})_{1\leq i,j\leq L}$$

to be the $L \times L$ matrix where r_{ij} is the number of j's in row i of T. $R(T)$ encodes T. As pointed out in (a preprint version of) [10], T might have exponentially many (in L) boxes, whereas $R(T)$ is a $O(L^2)$ description of T.

Example 5.8. If $\lambda = (4, 3, 1, 0, 0)$ and

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \longleftrightarrow R(T) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $X_{\alpha,w} = \{(S, R(T))\} \subseteq X$ such that the following hold:

(X.1′) $S \in \mathsf{Trans}(w, v)$,
(X.2′) $T \in \mathsf{SSYT}(\lambda(v), \phi(v))$, and
(X.3′) $\mathsf{delwt}(S) + \mathsf{content}(T) = \alpha$.

Proposition 5.9. $c_{\alpha,w} = \#X_{\alpha,w}$.

Proof. Iterating (23),

$$\mathfrak{S}_w = \sum_{\text{vexillary } v\in S_\infty} \sum_{S\in \mathsf{Trans}(w,v)} x^{\mathsf{delwt}(S)} \mathfrak{S}_v.$$

Hence

$$c_{\alpha,w} = \sum_{\text{vexillary } v\in S_\infty} \sum_{S\in \mathsf{Trans}(w,v)} [x^\alpha] x^{\mathsf{delwt}(S)} \mathfrak{S}_v. \tag{24}$$

The result then follows from by (21), (22), and (24) combined. □

Proposition 5.10 (cf. [6]). *Let* $\mathsf{code}(w) = (c_1, \ldots, c_L)$. *Suppose* $D(w')$ *is obtained from* $D(w)$ *using move* (T.1) *and* $D(w'')$ *is obtained from* $D(w)$ *with move* (T.2) *for a pivot in row* i. *There is an* $O(L^2)$*-time algorithm to compute*

(I) $\mathsf{code}(w') = (c_1, \ldots, c_{r-1}, c_r - 1, c_{r+1}, \ldots, c_L)$ *and*
(II) $\mathsf{code}(w'') = (c_1, \ldots, c_{i-1}, c_i + b, c_{i+1}, \ldots, c_{r-1}, c_r - b, c_{r+1}, \ldots, c_L)$, *for some* $b \in \mathbb{Z}_{>0}$.

Proof. By Proposition 3.2, determine $\mathbf{z}_w := (r, c)$ in $O(L^2)$-time.

For (I), $D(w')$ is obtained from $D(w)$ by deleting $\mathbf{z}_w$; so the expression in (I) is clear.

For (II), using Proposition 3.1, we can find $\mathbf{x} = (i, w(i))$, in $O(L^2)$-time; this is our (T.2) pivot. Notice that row r of $D(w) \cap \mathcal{R}$ is nonempty (it contains $\mathbf{z}_w = (r, c)$); let b be the number of boxes in this row. It is straightforward from the graphical description of $\mathcal{R}$ in terms of Rothe diagrams that each row of $D(w) \cap \mathcal{R}$ either has zero boxes or $b > 0$ boxes. Moreover, the d-th box (say, from the left) of each row are in the same column.

Suppose $j_1, \ldots, j_m \in [i+1, r]$ index the rows where $D(w) \cap \mathcal{R} \neq \emptyset$ (and thus has b boxes). (T.2) moves the b boxes of j_1 to row i and moves the b boxes of j_q to row j_{q-1} for $q = 2, \ldots, m$. As explained above $j_m = r$, so (T.2) moves no boxes into row r. Thus row r of $D(w'') \cap \mathcal{R}$ has zero boxes.

It remains to compute b in $O(L^2)$-time. Using Proposition 3.1 compute, in $O(L^2)$-time,

$$m := \#\{h < r \ : \ w(h) < w(i)\}.$$

Clearly $b = c_r - [(w(i) - 1) - m]$. □

Let $s_t = (x_r, m_t)$, as in (X.1), be a valid (multi)-deletion move on $u \in \mathcal{T}(w)$. Let $u^{\langle m \rangle} \in \mathcal{T}(w)$ be defined by $u \xrightarrow{x_k} \bullet \cdots \bullet \xrightarrow{x_k} u^{\langle m_t \rangle}$ (m_t-times).

Proposition 5.11. *Suppose* $u \in \mathcal{T}(w)$ *where* $\mathsf{code}(u) = (\widetilde{c}_1, \ldots, \widetilde{c}_{L'})$. *Let* $s_t = (x_k, m_t)$ *or* $s_t = i$ *be as in (X.1). Given input* $\mathsf{code}(u)$ *and* s_t, *there is an* $O(L^2)$ *algorithm to respectively determine if* $u \xrightarrow{x_k} \bullet \cdots \bullet \xrightarrow{x_k} u^{\langle m_t \rangle}$ *(m_t-times) or* $u \xrightarrow{i} u''$ *occurs in* $\mathcal{T}(w)$ *and (if yes) to compute*

- $\mathsf{code}(u^{\langle m_t \rangle})$ *in the case* $s_t = (x_k, m_t)$ *(a multi-deletion move (T.1)), or*
- $\mathsf{code}(u'')$ *in the case* $s_t = i$ *(a march move (T.2)).*

Proof. By Proposition 5.10, $L' \leq L$. Thus in our run-time analysis, we replace L' by L.

Proposition 3.2 finds $\mathbf{z}_u := (r, c)$ (or determines it does not exist) in $O(L^2)$-time. If $\mathbf{z}_u$ does not exist then u is dominant and thus vexillary; output s_t is invalid. Thus we assume henceforth that $\mathbf{z}_u$ exists.

Case 1: ($s_t = (x_k, m_t)$.) Proposition 3.1 finds $u(1), \ldots, u(L')$ in $O(L^2)$-time. Determine (taking $O(L^2)$ time) if

$$c_r - \left(\left(\min_{i\in[r]} u(i) \right) - 1 \right) \geq m_t, \tag{25}$$

holds. We claim that s_t is valid if and only if (25) holds and $k = r$. Indeed, observe

$$\#\{\text{boxes in row } r \text{ of } \mathsf{Dom}(u)\} = \left(\min_{i\in[r]} u(i)\right) - 1. \tag{26}$$

Thus, (25) is equivalent to the existence of m_t boxes in row r of $D(u) \setminus \mathsf{Dom}(u)$. By (T.1), if $k = r$ this is equivalent to being able to apply $\bullet \xrightarrow{x_r} \bullet$ successively m_t-times.

Finally, if s_t is valid, by m_t applications of Proposition 5.10 (I),

$$\mathsf{code}(u^{\langle m_t\rangle}) = (\widetilde{c}_1, \ldots, \widetilde{c}_{r-1}, \widetilde{c}_r - m_t, \widetilde{c}_{r+1}, \ldots, \widetilde{c}_{L'}). \tag{27}$$

Hence we can output (27) in $O(L^2)$-time.

Case 2: ($s_t = i$.) By Proposition 3.1, determine $u(1), \ldots, u(L')$ from $\mathsf{code}(u)$ in $O(L^2)$-time. In particular this computes $\mathbf{x} := (i, u(i))$ in $O(L^2)$-time. To decide if s_t is valid we must determine if $\mathbf{x} \in \mathsf{Piv}(\mathbf{z}_u)$. To do this, first calculate (in $O(L)$-time)

$$u_{NW}(\mathbf{z}_u) := \{(j, u(j)) \ : \ j < r, u(j) < c\}.$$

By definition,

$$\mathsf{Piv}(\mathbf{z}_u) = \{(j, u(j)) \in u_{NW}(\mathbf{z}_u) \ : \ \nexists (h, u(h)) \in u_{NW}(\mathbf{z}_u) \text{ with } h > j, u(h) > u(j)\}.$$

$\mathsf{Piv}(\mathbf{z}_u)$ takes $O(L)$-time to compute since $\#u_{NW}(\mathbf{z}_u) \leq r - 1 \leq L - 1$. Hence we check if $\mathbf{x} \in \mathsf{Piv}(\mathbf{z}_u)$ in $O(L)$-time. If this is false, we output a rejection. Otherwise, Proposition 5.10 outputs $\mathsf{code}(u'')$ in $O(L^2)$-time. □

Proposition 5.12. *If $S = (s_1, \ldots, s_h) \in \mathsf{Trans}(w, v)$ then $h \leq L^2$.*

Proof. Let $w := w_0 \xrightarrow{s_1} w_1 \xrightarrow{s_2} \ldots \xrightarrow{s_h} w_h = v$ be the path in $\mathcal{T}(w)$ associated to S. By (T.1) and (T.2), $\mathbf{z}_{w_{t+1}}$ is weakly northwest of $\mathbf{z}_{w_t}$. Hence, for any fixed r, those $t \in [0, h-1]$ with $\mathbf{z}_{w_t}$ in row r form an interval $I^{(r)} \subseteq [0, h-1]$. Since $1 \leq r \leq L$, it suffices to prove

$$\#I^{(r)} \leq 2(r-1). \tag{28}$$

By (X.1) the transition moves acting on row r alternate between multi-(T.1) moves (x_r, m_t) and (T.2) moves. Thus to show (28), it is enough to prove

$$\#\{t \in I^{(r)} : w_{t-1} \to w_t \text{ is a (T.2) move}\} \leq r - 1. \tag{29}$$

Consider a march move i with $\mathbf{z}_{w_{t-1}} = (r, c)$ and $\mathbf{x} = (i, w_{t-1}(i)) \in \mathsf{Piv}(\mathbf{z}_{w_{t-1}})$. By (T.2), if $(r, c') \in D(w_{t-1})$ is in the same connected component as $\mathbf{z}_{w_{t-1}}$, the move i takes (r, c') strictly north of row r. Thus, each march move strictly reduces the number of components in row r. Let $t_0 = \min\{t \in I^{(r)}\}$. Since there are at most r $\bullet$'s weakly above row r, $D(w_{t_0})$ has at most $r - 1$ (non-dominant) components in row r. Hence (29) holds, as desired. □

Proposition 5.13. *Let v be vexillary with $\mathsf{code}(v) = (c_1, \ldots, c_{L'})$ and $L' \leq L$. There exists an $O(L^2)$-time algorithm to check if $R = (r_{ij})_{1\leq i,j\leq L'}$ is $R = R(T)$ for some $T \in \mathsf{SSYT}(\lambda(v), \phi(v))$.*

Proof. Since $L' \leq L$, it is $O(L^2)$-time to calculate $\phi(v), \lambda(v)$. Let

$$\lambda_i := \sum_{j=1}^{L'} r_{ij}, \text{ for } 1 \leq i \leq L'.$$

First verify (in $O(L)$-time) that $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i \leq L' - 1$. Then $R = R(T)$ where T is the (unique) row weakly increasing tableau of shape λ with r_{ij} many j's in row i.

To verify $T \in \mathsf{SSYT}(\lambda(v), \phi(v))$ we must check that it is (i) is flagged by $\phi(v)$, (ii) has shape $\lambda(v)$, and (iii) is semistandard. For (i), we need

$$r_{ij} = 0 \text{ if } j > \phi(v)_i, \text{ for all } i, j \in [L']. \tag{30}$$

For (ii), we need

$$\lambda_i = \lambda(v)_i \text{ for each } i \in [L']. \tag{31}$$

For (iii), it remains to ensure that T is column strict, i.e.,

$$\sum_{j' \le j} r_{i+1,j'} \le \sum_{j' < j} r_{i,j'} \text{ for each } i \in [L'-1], j \in [L']. \tag{32}$$

We found the inequalities (31) and (32) from a (preprint) version of [10]. The inequalities (30), (31), and (32) can be checked in $O(L^2)$-time since $i, j \in [L'] \subseteq [L]$. □

The following completes our proof that we can check that $(S, R) \in X_{\alpha,w}$ in $L^{O(1)}$-time.

Proposition 5.14. *Given $(S, R) \in X$ and $(\mathsf{code}(w), \alpha)$, one can determine if $(S, R) \in X_{\alpha,w}$ in $L^{O(1)}$-time.*

Proof. By Propositions 5.11 and 5.12 combined, one determines in $O(L^4)$-time if S encodes a path $w := w_0 \xrightarrow{s_1} w_1 \xrightarrow{s_2} \cdots \xrightarrow{s_h} w_h = v$ in $\mathcal{T}(w)$. If so, the length of $\mathsf{code}(v)$ is at most L. Thus, using Theorem 5.1, one checks v is vexillary in $O(L^3)$-time. This decides if S satisfies (X.1′). Proposition 5.13 checks R satisfies (X.2′) in $O(L^2)$-time. Finally since $h \le L^2$, computing $\mathsf{delwt}(S)$ takes $O(L^2)$-time. Hence (X.3′) is checkable in $O(L^2)$ time. □

Proof of Theorem 5.6: By Proposition 5.9, $\#X_{\alpha,w} = c_{\alpha,w}$. By Proposition 5.12, $(S, R) \in \#X_{\alpha,w}$ only if the list S has at most L^2 elements. Assuming this, we check (S, R) satisfies (X.1) and (X.2) in $O(L^2)$-time. Using Proposition 5.14, we can verify $(S, R) \in X_{\alpha,w}$ in $L^{O(1)}$-time. Thus, given input α and $\mathsf{code}(w)$, computing $c_{\alpha,w}$ is in $\#\mathsf{P}$. □

5.4. **Hardness, and the conclusion of the proof of Theorem 1.5.** *Schur polynomials* are an important basis of the vector space of symmetric polynomials. The Schur polynomial $s_\lambda = a_{\lambda+\delta}/a_\delta$ where $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$, $a_\gamma := \det(x_i^{\gamma_j})_{i,j=1}^n$, and $\delta = (n-1, n-2, \ldots, 2, 1, 0)$. The flagged Schur function of Section 5.1 is a generalization of the Schur polynomial.

A permutation w is *grassmannian* if it has at most one *descent* i, i.e., where $w(i) > w(i+1)$. Given a partition $(\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_L \ge 0)$ define a grassmannian permutation w_λ by setting

$$w_\lambda(i) = i + \lambda_{L-i+1} \text{ for } 1 \le i \le L.$$

For w_λ grassmannian, it is well-known (see, e.g., [8]) that

$$\mathsf{code}(w_\lambda) = (\lambda_L, \lambda_{L-1}, \ldots, \lambda_1). \tag{33}$$

Moreover,

$$\mathfrak{S}_{w_\lambda} = s_\lambda(x_1, \ldots, x_L) = \sum_{\alpha \in \mathbb{Z}_{\ge 0}^L} K_{\lambda,\alpha} x^\alpha, \tag{34}$$

where $K_{\lambda,\alpha}$ is the *Kostka coefficient*. This number counts semistandard tableaux of shape λ with content α.

By (34),

$$c_{\alpha,w_\lambda} = K_{\lambda,\alpha}. \tag{35}$$

By Theorem 5.6, counting $c_{\alpha,w}$ is in $\#\mathsf{P}$. Suppose there is an oracle to compute $c_{\alpha,w}$ in polynomial time in the input length of $(\mathsf{code}(w), \alpha)$. This input length is the same as for the input λ, α for $K_{\lambda,\alpha}$. Hence (33) and (35) combined imply a polynomial-time counting reduction from $\{c_{\alpha,w}\}$ to Kostka coefficients. Now H. Narayanan [10] proved that counting $K_{\lambda,\alpha}$ is a $\#\mathsf{P}$-complete problem. Thus counting $c_{\alpha,w}$ is a $\#\mathsf{P}$-complete problem. □

Remark 5.15. Suppose the input for counting $c_{\alpha,w}$ is (α, w) where $w \in S_n$ (in one-line notation). Then the above counting reduction is not polynomial time in the input length of the Kostka problem. For example, suppose $\lambda = \alpha = (2^L, 2^L, \ldots, 2^L)$ (L-many). Then the input length of this instance of the Kostka problem is $2L^2 \in O(L^2)$. On the other hand, $w_\lambda \in S_{L+2^L}$. Therefore, a polynomial time algorithm for the Schubert coefficient problem in n would have $\Omega(2^L)$ run time for the Kostka problem.

It seems unlikely that there is a polynomial-time reduction under this input assumption. This is our justification to encode w via $\mathsf{code}(w)$ rather than one line notation. □

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UCLA, Los Angeles, CA 90095
Email address: aadve@g.ucla.edu

Dept. of Mathematics, U. Illinois at Urbana-Champaign, Urbana, IL 61801, USA
Email address: cer2@illinois.edu, ayong@illinois.edu