Presentation on Hecke Algebra and Kazhdan-Lusztig Polynomials

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Abstract

This note is part of the ICLUE Summer 2022 program, where I gave a lecture on an introduction to the Hecke Algebra and Kazhdan-Lusztig polynomials.

1 Coxeter System

Definition 1.1 (Coxeter System)**.** Formally, a Coxeter group can be defined as a group with the presentation

$$\langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where the coefficients m_{ij} need to satisfy the following conditions:

1. $m_{ii} = 1$ for all i.
2. $m_{ij} \geq 2$ for $i \neq j$
3. $m_{ij} = \infty$ means no relation of the form $(s_i s_j)^m$ should be imposed.

The pair (W, S) where W is a Coxeter group with generators $S = \{r_1, \ldots, r_n\}$ is called a **Coxeter system**.

Several observations can be made immediately from this definition:

1. The generators s_i's are all reflections (involutions).
2. $m_{ij} = m_{ji}$ for any i, j. This is because, if $(xy)^m = 1$, then

$$(yx)^m = y(xy)^m y = yy = 1.$$

Example 1.2. As expected, the symmetric group S_n is isomorphic to the Coxeter group W made of $\{s_1, \cdots, s_{n-1}\}$ and the relations defined by $(s_i s_{i+1})^3 = 1$. In order words, $m_{ij} = 1$ when $i = j$; $m_{ij} = 3$ when $|i - j| = 1$, and 0 otherwise.

Definition 1.3 (Length Function)**.** Since the generators $s \in S$ have order 2 in W, each $w \neq 1$ in W can be written in the form $w = s_1 s_2 \cdots s_r$ for some s_i (not necessarily distinct) in S. If r is as small as possible, call it the **length** of w, written $\ell(w)$, and call any expression of w as a product of r elements of S a **reduced expression**. By convention, $\ell(1) = 0$.

Proposition 1.4. *Some elementary properties of the length function.*

1. $\ell(w) = \ell(w^{-1})$.
2. $\ell(ww') \leq \ell(w) + \ell(w')$
3. $\ell(ww') \geq \ell(w) - \ell(w')$
4. $\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$.

Proof. (1) is immediate. (2) follows from the possible cancelation made when concatenating w and w'. (3) follows from applying (2) at the pairs ww' and $(w')^{-1}$. (4) follows from applying (2) and (3) at $w' = s$. □

Definition 1.5 (Reflection)**.** Let (W, S) be a Coxeter system. The set of reflections T are defined to be $T = \{wsw^{-1} \mid w \in W, s \in S\}$.

Remark. The definition is motivated by the following facts.

1. For any $t \in T$, we have $t^2 = 1$.
2. Clearly $S \subset T$, and we sometimes call S the set of simple reflections.
3. In the case of symmetric groups, S is the set of adjacent transpositions. Since

$$\sigma(i, i+1)\sigma^{-1} = (\sigma(i), \sigma(i+1)),$$

it follows that T in S_n is the set of all transpositions.

Now it's clear why we call T the set of reflections.

The following theorem, which we state without proof, is at the heart of the Coxeter system.

Theorem 1.6 (Exchange Conditions)**.** *Let $w = s_1 \cdots s_r$ $(s_i \in S)$, not necessarily a reduced expression. Suppose a reflection $t \in T$ satisfies $\ell(wt) < \ell(w)$. Then there is an index i for which $wt = s_1 \cdots \widehat{s_i} \cdots s_r$ (omitting s_i). If the expression for w is reduced, then i is unique.*

2 Bruhat Order

Among the possible ways to partially order W in a way compatible with the length function, the most useful has proven to be the Bruhat ordering, defined as follows. Recall that a partial order can be converted to a directed graph, so we shall start from there.

Definition 2.1. Let T be the set of reflections in W with respect to roots. Write $w' \to w$ if $w = w't$ for some $t \in T$ with $\ell(w) > \ell(w')$. Then define $w' < w$ if there is a sequence $w' = w_0 \to w_1 \to \ldots \to w_m = w$. It is clear that the resulting relation $w' \leq w$ is a partial ordering of W (reflexive, antisymmetric, transitive), with 1 as the unique minimal element.

Remark (1). The reason why the Bruhat order is a partial order is the following.

- (Reflexive): Obvious.
- (Transitive): Suppose $w_1 \leq w_2$ and $w_2 \leq w_3$. If any of the two $\leq$ is equality, then there's nothing to prove. If not, then there exists a chain $w_1 \to \cdots \to w_2 \to \cdots w_3$, and $\ell(w_1) < \ell(w_3))$, so $w_1 \leq w_3$.
- (Symmetric): Suppose $w_1 \leq w_2$ and $w_2 \leq w_1$. If any of the two $\leq$ is not equality, say the first one, then $\ell(w_1) < \ell(w_2) \leq \ell(w_1)$, which is a contradiction, so $w_1 = w_2$.

Remark (2). The relation "$\to$" can be used to generate a directed graph, called the Bruhat Graph. The Bruhat order is obtained from this graph by "transitive closure," i.e., throwing in all the missing relations that come from transitivity.

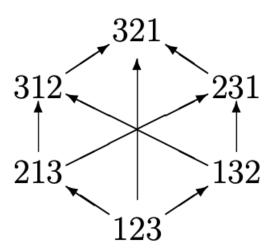

Figure 1: The Bruhat graph of S_3.

We need the following proposition to prove an equivalent definition of Bruhat order.

Proposition 2.2. *Let $w' \leq w$ and $s \in S$. Then either $w's \leq w$ or else $w's \leq ws$ (or both).*

Theorem 2.3 (Subexpression)**.** *Let $w = s_1 \cdots s_r$ be a fixed, but arbitrary, reduced expression for w. Then $w' \leq w$ if and only if w' can be obtained as a subexpression of this reduced expression.*

Proof. Let us first show that any $w' < w$ occurs as a subexpression of the given reduced expression for w. Start with the case $w' \to w$, say $w = w't$. Since $\ell(w') < \ell(w)$, the Strong Exchange Condition can be applied to the pair t, w to yield $w' = wt = s_1 \cdots \widehat{s}_i \cdots s_r$ for some i. This argument can be iterated, and so w' must be a subexpression of (any) reduced expression of w.

In the other direction, we are given a subexpression $w' = s_{i_1} \cdots s_{i_q}$ of w and must show it to be $\leq w$. Here we can use induction on $r = \ell(w)$, the case $r = 0$ being trivial. If $i_q < r$, then w' is a subexpression of $s_1 \cdots s_{r-1}$, so by the induction hypothesis:

$$s_{i_1} \cdots s_{i_q} \leq s_1 \cdots s_{r-1} = ws_r < w.$$

On the other hand, if $i_q = r$ we first use induction to get $s_{i_1} \cdots s_{i_{q-1}} \leq s_1 \cdots s_{r-1}$, and then apply Proposition 2.2 to get either

$$s_{i_1} s_{i_2} \cdots s_{i_q} \leq s_1 \cdots s_{r-1} < w \qquad \text{or} \qquad s_{i_1} s_{i_2} \cdots s_{i_q} \leq s_1 s_2 \cdots s_r = w.$$

In either case, $w' < w$. ∎

Remark. Theorem 2.3 is an extremely important characterization of Bruhat order, though it might seem implausible at first because of its independence from the choice of reduced words of w. Transitivity would be far from obvious. But it helps to make explicit computations more transparent.

3 Generic Algebra and Hecke Algebra

Let A be a commutative ring with 1. Let (W, S) be a Coxeter system. We shall begin with a very general construction of associative algebras over a commutative ring A. Such an algebra will have a free A-basis parametrized by the elements of W, together with a multiplication law which reflects in a certain way the multiplication in W. The algebra will also depend on some parameters $a_s, b_s \in A(s \in S)$, subject only to the requirement that $a_s = a_t$ and $b_s = b_t$ whenever s and t are conjugate in W.

The starting point for the construction is a free A-module $\mathcal{E}$ on the set W, with basis elements denoted T_w $(w \in W)$.

Definition 3.1 (Generic Algebra)**.** Given elements a_s, b_s as above, there exists a unique structure of associative A-algebra on the free A-module $\mathcal{E}$, with T_1 acting as the identity (check), such that the following conditions hold for all $s \in S, w \in W$:

$$T_sT_w = T_{sw} \qquad \text{if } \ell(sw) > \ell(w) \tag{1}$$

$$T_sT_w = a_sT_w + b_sT_{sw} \qquad \text{if } \ell(sw) < \ell(w) \tag{2}$$

The algebra described by the theorem, denoted $\mathcal{E}_A(a_s, b_s)$, will be called a generic algebra.

Remark. Many familiar structures can be obtained from the generic algebra. For example, let $a_s = 0$ and $b_s = 1$, then we have the group algebra $A[W]$. Another choice of parameters leads to a "Hecke algebra", which will be defined later.

Next we will focus on an equivalent definition of the generic algebra.

Proposition 3.2. *The two conditions in Definition 3.1 are equivalent to the following:*

$$T_sT_w = T_{sw} \qquad \textit{if } \ell(sw) > \ell(w) \tag{3}$$

$$T_s^2 = a_sT_s + b_sT_1 \qquad \textit{if } \ell(sw) < \ell(w) \tag{4}$$

Proof. (3)(4) are clearly consequences of (1)(2). To obtain (2) from (3)(4), suppose $\ell(sw) < \ell(w)$. Since $w = ssw$, we have $\ell(sw) < \ell(s(sw))$. Therefore, apply (3) to obtain

$$T_sT_{sw} = T_{ssw} = T_w.$$

Now, multiply both sides by T_s and apply (4) to obtain

$$\begin{aligned} T_sT_w &= (a_sT_s + b_sT_1)T_{sw} \\ &= a_sT_sT_{sw} + b_sT_{sw} \\ &= a_sT_w + b_sT_{sw}, \end{aligned}$$

which is exactly (2). ☐

Having constructed generic algebras $\mathcal{E}_A(a_s, b_s)$ over an arbitrary commutative ring A, we now make a special choice to obtain the Hecke Algebra.

Definition 3.3 (Hecke Algebra)**.** Let A be the ring $\mathbb{Z}\left[q, q^{-1}\right]$ of Laurent polynomials over $\mathbb{Z}$ in the indeterminate q. With the further convention that $a_s = q-1$ and $b_s = q$ for all $s \in S$, we write $\mathcal{H}$ for the resulting generic algebra and call it the Hecke algebra of W.

The relations (3) and (4) now become:

$$\begin{aligned} T_s T_w &= T_{sw} && \text{if } \ \ell(sw) > \ell(w) \\ T_s^2 &= (q-1)T_s + qT_1 && \text{if } \ \ell(sw) < \ell(w) \end{aligned}$$

It should be noted that all the conditions we listed here are one-sided, i.e. s always appears on the left of w. However, one can easily prove by induction on $\ell(w)$ that the right-handed version of these conditions are true as well.

The first special feature to notice in $\mathcal{H}$ is the existence of inverses for the basis elements T_w, because of the presence of q^{-1}. Indeed, the relations imply that for all $s \in S$:

$$T_s^{-1} = q^{-1}T_s - \left(1 - q^{-1}\right) T_1.$$

If $w = s_1 \cdots s_r$ (reduced expression), we know that $T_w = T_{s_1} \cdots T_{s_r}$. Therefore every T_w is invertible in $\mathcal{H}$. However, as $\ell(w)$ increases it will be progressively more complicated to work out the inverse explicitly as a linear combination of the canonical basis of $\mathcal{H}$. What we can do in this direction introduces an important family of polynomials (the "R polynomials").

4 *R*-polynomials

Theorem 4.1. *For all $w \in W$,*

$$\left(T_{w^{-1}}\right)^{-1} = \varepsilon_w q_w^{-1} \sum_{x \le w} \varepsilon_x R_{x,w}(q) T_x,$$

where $R_{x,w}(q) \in \mathbb{Z}[q]$ is a polynomial of degree $\ell(w) - \ell(x)$ in q, and where $R_{w,w}(q) = 1$. Here ε_w is defined to be $(-1)^{\ell(w)}$, and q_w is defined to be $q^{\ell(w)}$.

Remark. While we will not cover the entire (complicated) proof, we shall verify two special cases: the statement is clear when $w = 1$, as $\varepsilon_w = (-1)^0 = 1$, $q_w = q^0 = 1$, and so

$$\varepsilon_w q_w^{-1} \sum_{x \le w} \varepsilon_x R_{x,w}(q) T_x = \sum_{x \le w} R_{x,w}(q) T_x = T_1.$$

The statement is also clear if $w = s \in S$. We have $\left(T_{s^{-1}}\right)^{-1} = T_s^{-1}$, and $\varepsilon_s = -1$, $q_s = q$. If we set $R_{1,s} = q - 1$, then $x \le s$ if and only if $x = 1$ or s. Therefore,

$$\begin{aligned} \varepsilon_s q_s^{-1} \sum_{x \le s} \varepsilon_x R_{x,s}(q) T_x &= (-1) q^{-1} \left[R_{1,s}(q) T_1 - R_{s,s}(q) T_s\right] \\ &= -q^{-1} \left[(q-1) T_1 - T_s\right] \\ &= q^{-1} T_s - \left(1 - q^{-1}\right) T_1 \\ &= T_s^{-1}. \end{aligned}$$

The general statement is obtained by induction on $\ell(w)$. For convenience, define $R_{x,w}$ to be 0 whenever $x \not\leq w$.

There is an explicit algorithm for computing $R_{x,w}$ recursively, starting with the fact that $R_{w,w} = 1$ for all $w \in W$, while $R_{x,w} = 0$ unless $x \leq w$. For the induction step, we need to compute $R_{x,w}$, assuming that all polynomials $R_{y,z}$ are known for $\ell(z) < \ell(w)$. Fix $s \in S$ for which $sw < w$. Then two configurations have to be dealt with, as in Lemma 7.4 :

(A) $x < w, sx < x$ (forcing $sx < sw$). Here we found that $R_{x,w} = R_{sx,sw}$, which is already known since $sw < w$.

(B) $x < w, x < sx$ (forcing $sx \leq w$ and $x \leq sw$). Here we found that $R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$, both terms of which are already known. (Recall that the first term has degree $\ell(w) - \ell(x)$, while the second term has lower degree and might be 0 .)

5 Kazhdan-Lusztig Polynomials

Now we begin to introduce a little bit about Kazhdan-Lusztig polynomials, which will be covered in greater detail in another lecture.

Definition 5.1. Let $\iota : \mathcal{H} \to \mathcal{H}$ be the involution map that sends q to q^{-1} and T_w to $(T_{w^{-1}})^{-1}$.

We need to verify that ι is an involution. First we check that it $\iota^2(T_s) = T_s$ for all $s \in S$.

$$\iota(T_s) = T_s^{-1} = q^{-1}T_s - \left(1 - q^{-1}\right) T_1$$

$$\begin{aligned}\iota^2(T_s) &= q\left[q^{-1}T_s - \left(1 - q^{-1}\right) T_1\right] - (1-q)T_1 \\ &= T_s - (q-1)T_1 + (q-1)T_1 \\ &= T_s.\end{aligned}$$

Since $\mathcal{H}$ is generated by $T_s : s \in S$, it remains to be shown that $\iota : \mathcal{H} \to \mathcal{H}$ is a ring homomorphism, i.e. $\iota(T_{w'w}) = \iota(T_{w'})\iota(T_w)$. To this end, one can first show that $\iota(T_s)\iota(T_w) = \iota(T_{sw})$ for all $s \in S$ and $w \in W$, and then use induction to obtain the claim.

The involution ι will play a key role in the definition of Kazhdan-Lusztig polynomials. We now look for a new basis $\{C_w\}$ of the A-module $\mathcal{H}$, indexed again by W, but consisting of elements **fixed by the involution ι**.

We can see how to get started by experimenting with the following formula :

$$T_s^{-1} = q^{-1}T_s - \left(1 - q^{-1}\right) T_1.$$

It is easy to check that ι sends $T_s - qT_1$ to $q^{-1}\left(T_s - qT_1\right)$, as

$$\begin{aligned}\iota(T_s - qT_1) &= T_s^{-1} - q^{-1}T_1 \\ &= q^{-1}T_s - \left(1 - q^{-1}\right) T_1 - q^{-1}T_1 \\ &= q^{-1}T_s - T_1 \\ &= q^{-1}(T_s - qT_1).\end{aligned}$$

Unfortunately this is not fixed by ι. If we are willing to introduce a square root of q (written $q^{\frac{1}{2}}$), we therefore have an element fixed by ι for each $s \in S$ (check):

$$C_s := q^{-\frac{1}{2}}\left(T_s - qT_1\right).$$

Formally, we replace $\mathbb{Z}[q, q^{-1}]$ by the ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ of Laurent polynomials in the indeterminate $q^{\frac{1}{2}}$, so that the previous ring A becomes a subring of the new one. This has no effect on the previous formal calculations in $\mathcal{H}$.

It is tempting to construct further ι-invariants simply by multiplying various $C_s (s \in S)$, in the spirit of the way the original basis elements T_w of $\mathcal{H}$ are built out of the T_s. This, however, has two problems:

1. The labeling may be confusing. For example, $C_sC_tC_s \neq C_tC_sC_t$ when $sts = tst$, so we can't just label the element to be C_{sts}.

2. The polynomials arising as the coefficients are too complicated.

What we are seeking in general is an ι-invariant element C_w which is a linear combination of the T_x for $x \leq w$ (the coefficient of T_w being nonzero) and whose polynomial coefficients are as uncomplicated as possible. The following basic theorem of Kazhdan-Lusztig [1] provides an optimal choice:

Theorem 5.2. *For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}$ having the following two properties:*

(a) $\iota\left(C_w\right) = C_w$

(b) $C_w = \varepsilon_w q_w^{\frac{1}{2}} \sum_{x \leq w} \varepsilon_x q_x^{-1} \overline{P}_{x,w} T_x$, *where* $P_{w,w} = 1$ *and* $P_{x,w}(q) \in \mathbb{Z}[q]$ *has degree* $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$ *if* $x < w$.

The polynomials $P_{x,w}$ turn out to be of fundamental interest. They are called the Kazhdan-Lusztig polynomials. They are appreciably more subtle than the earlier $R_{x,w}$. For example, their precise degrees are not readily predictable. It is conjectured in Kazhdan-Lusztig that all coefficients of $P_{x,w}$ are nonnegative, but this remains unproved (at the time of writing) except in some important special cases.

References

[1] Humphreys (1992) *Reflection Groups and Coxeter Groups.*

[2] Francesco Brenti et al. *Kazhdan-Lusztig polynomials: History Problems, and Combinatorial Invariance.*