05/18/22 Notes

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- Next meeting time: 10am on Friday at AH347

1 Group Rings and G-Modules

Recall, a representation of a group G in a vector space V over $\mathbb{F}$ is a homomorphism $f : G \to \mathrm{GL}(V)$. We can change our perspective and think of this as scalar multiplication on V by letting $g \cdot \vec{v} \mapsto [f(g)](\vec{v})$.

Definition 1.1 (group ring)**.** The group ring $\mathbb{F}[G]$ is the set of all finite formal linear combinations of group elements:

$$\mathbb{F}[G] = \left\{ \sum_{i=1}^{n} a_i g_i \,\middle|\, a_i \in \mathbb{F},\, g_i \in G,\, n \in \mathbb{Z}^+ \right\}$$

$\mathbb{F}[G]$ is a vector space over $\mathbb{F}$, and $\mathbb{F}[G]$ is a ring with multiplication as in a polynomial, simplifying with the group operation if necessary. For example: $(ag_1+bg_2)\cdot(cg_1+dg_2) = ac(g_1\cdot g_1)+ad(g_1\cdot g_2)+bc(g_2\cdot g_1)+bd(g_2\cdot g_2)$.

Definition 1.2 (module)**.** For a ring R, a (left) R-module is a set M with two operations $+ : M \times M \to M$ and $\cdot : R \times M \to M$ such that $(M, +)$ is an abelian group and for all $r, s \in R$ and $x, y \in M$:

1. $r \cdot (x + y) = r \cdot x + r \cdot y$
2. $(r + s) \cdot x = r \cdot x + s \cdot x$
3. $(r \cdot s) \cdot x = r \cdot (s \cdot x)$
4. $1 \cdot x = x$

Then, for a given representation $f : G \to \mathrm{GL}(V)$, it follows that V is an $\mathbb{F}[G]$-module, with scalar multiplication extended naturally from the $g \cdot \vec{v}$ defined above. For short we simply say G-module.

Proposition 1.3. *V is a G-module $\iff$ There exists a representation $f : G \to \mathrm{GL}(V)$.*

Proof. ($\implies$) Suppose V is a G-module. Define $f : G \to \mathrm{GL}(V)$ by $f(g) : \vec{v} \mapsto g \cdot \vec{v}$. This is an invertible linear map since $g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2$ and $g^{-1} \cdot (g \cdot v) = (g^{-1} \cdot g) \cdot v = 1 \cdot v = v$ by the definition of a module. f is a homomorphism because $(g_1 \cdot g_2) \cdot v = g_1 \cdot (g_2 \cdot v)$. Hence, f is a representation.

($\impliedby$) Suppose $f : G \to \mathrm{GL}(V)$ is a representation. Define scalar multiplication by:

$$(a_1 g_1 + \cdots + a_n g_n) \cdot v = a_1 \cdot [f(g_1)](v) + \cdots + a_n \cdot [f(g_n)](v)$$

The first property follows from the linearity of $f(g)$. The second property follows directly from the definition. The third and fourth properties follow from the properties of a homomorphism since $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$ and $f(1) = \mathrm{id}_V$. Hence, V is a G-module. □

Definition 1.4 (G-submodule)**.** A subset $W \subseteq V$ of a G-module is a G-submodule if it is closed under addition and multiplication by elements of G.

A representation is a subrepresentation of V if it is a G-submodule of V as a G-module. Every G-module V has the submodules V and $\{0\}$. These are called the trivial submodules.

2 More Examples of Representations

Definition 2.1 (degree)**.** The degree of a representation $f : G \to \mathrm{GL}(V)$ is the dimension of V.

Example 2.2. $f : G \to \mathrm{GL}(V), f(g) = I_V$, or in terms of G-modules, $g \cdot v = v$, $\forall g \in G, v \in V$

The example above is known as the Trivial Representation

Example 2.3. $G = (\mathbb{Z}, +)$, $V = \mathbb{R}^3$, $f(n) = I_3 + nD + \frac{n(n+1)}{2}D^2$, where $D_{i,j} = 1$ if $j = i + 1$, and 0 otherwise.

That f is a group homomorphism can be seen from the fact that $f(n) = f(1)^n$, which can be shown by binomial expansion of $(I_3+D+D^2)^n$ and $(I_3-D)^n$ for $n \geq 0$, and noting that $(I_3+D+D^2)(I_3-D) = I_3-D^3$, $D^3 = 0$.

Example 2.4. $G = \{z \in \mathbb{C} : |z| = 1\}$, $f(z) = z^n$ for some $n \in \mathbb{Z}$

In fact these are all inequivalent irreducible representations of G, and form an exhaustive list of all the irreducible representations of G. Furthermore, all of these are one-dimensional.
But what does it mean for two representations to be inequivalent, or irreducible?

3 Irreducible Representations and Equivalence of Representations

Definition 3.1. A representation $f : G \to \mathrm{GL}(V)$ is irreducible if $\{0\}$ and V are the only subspaces of V that are invariant under $f(g)$ for every $g \in G$.

In terms of G-modules, a G-module V is an irreducible representation of G iff it has no nontrivial submodules

Example 3.2 (a reducible representation)**.** Let $\mathfrak{S}_3$ act on $\mathbb{C}^3$ by: $\sigma \cdot (v_1, v_2, v_3) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$. Then, $\mathbb{C}^3$ has two G-submodules given by $\{(v, v, v) \mid v \in \mathbb{C}\}$ and $\{(v_1, v_2, v_3) \in \mathbb{C}^3 \mid v_1 + v_2 + v_3 = 0\}$

Proposition 3.3. *A representation* $f : G \to \mathrm{GL}(V)$ *is irreducible* $\iff$ $\forall v \in V \setminus \{0\}$ *the vectors* $G \cdot v := \{f(g)v : g \in G\}$ *span* V.

Proof. ($\implies$) Since the span of $G \cdot v$ is a G-invariant subspace of V (Because $f(h)(f(g)v) = f(gh)v$), and it contains at least one nonzero vector, v, by irreducibility of f, it must be V itself.

($\impliedby$) If W is a G-invariant subspace of V, then either $W = 0$ or W contains at least one nonzero vector w. In the latter case, since W is a G-invariant subspace of V, $V = \mathrm{Span}(G \cdot w) \subseteq W \subseteq V$, and therefore W=V. □

Definition 3.4. If G be a group, V and W are vector spaces, and $f_1 : G \to \mathrm{GL}(V)$ and $f_2 : G \to \mathrm{GL}(W)$ are representations of G, then f_1 and f_2 are equivalent representations of G if there is some invertible linear map $T : V \to W$ such that $\forall g \in G,\ f_2(g) = T f_1(g) T^{-1}$.

In other words, f_1 and f_2 are equivalent representations of G if $\forall g \in G$, $f_2(g)$ is the same as $f_1(g)$ up to a change of basis, and the change of basis is the same for all g.
In terms of G-modules, two representations of G, V and W, are equivalent as G-modules iff there is some bijective module homomorphism $T : V \to W$ which preserves the action of G. That is, $T(g \cdot v) = g \cdot T(v)$.