Lie Groups & the Haar Measure

Dylan Roscow

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1 Introduction

1.1 Preliminaries

Definition 1.1. A *manifold* is a locally Euclidean Hausdorff space which has a countable basis.

Example 1.2. $\mathbb{R}^n$ itself (because it is trivially homeomorphic to itself).

Example 1.3. Any sphere S^n by taking a small open ball around the point.

Example 1.4. Any torus $T^n = (S^1)^n$ and more generally any product space $M = \prod_{i=1}^{N} M_i$

Example 1.5. An open subset of $\mathbb{R}^n$.

Example 1.6. $\mathrm{GL}_n(\mathbb{R})$ (associating matrices to points in $\mathbb{R}^{n^2}$) since it is an open subset.

Definition 1.7. A *coordinate chart* on an n-dimensional manifold is a pair (U, φ) consisting of an open set U of the manifold and a homeomorphism $\varphi : U \to V$ with an open set $V \subseteq \mathbb{R}^n$.
An *atlas* is a family of coordinate charts which covers the manifold.

If the domains of two coordinate charts overlap, we want to "think of" the codomains as overlapping. To do this, we define the transition functions by composing one chart with the inverse of the other.

Definition 1.8. A *smooth atlas* is an atlas whose transition functions are smooth (as functions $\mathbb{R}^n \to \mathbb{R}^n$).

We say two smooth atlases are *equivalent* if their union is also a smooth atlas. This gives an equivalence relation $\sim$ on smooth atlases.

Definition 1.9. Let $\mathrm{SA}(M)$ denoted the set of smooth atlases on M. Then, a *smooth structure* on M is an element of $\mathrm{SA}(M)/\sim$.

Definition 1.10. A *smooth manifold* $(M, \mathcal{A})$ is a manifold M together with a maximal smooth atlas $\mathcal{A}$. That is, a smooth atlas which is not a proper subset of any smooth atlas.

We can define a smooth manifold by defining a smooth atlas and let the smooth structure be its equivalence class.

Example 1.11 (Very Basic Example)**.** On $\mathbb{R}$ consider the atlas consisting of only the chart $(\mathbb{R}, \mathrm{id})$ with $\mathrm{id}(x) = x$.

We might hope that $\mathbb{R}$ would have only one equivalence class and thus a unique smooth structure, but un fortunately, $\mathbb{R}$ many different smooth structures. And, typically a manifold will have many distinct smooth structures.

Example 1.12 (Distinct)**.** $(\mathbb{R}, \varphi)$ with $\varphi(x) = x^3$ gives a distinct smooth structure on $\mathbb{R}$.

Sometimes the notion of "smooth" does not align with intuition.

Example 1.13 (Weird)**.** The graph of $|x|$ in $\mathbb{R}^2$ with the chart given by $(x, y) \mapsto x$.

An example with more than one chart

Example 1.14 (Circle)**.** S^1 by taking two overlapping intervals which cover it.

Similarly all the manifolds above are actually smooth manifolds. In fact, it is actually difficult to find a manifold that has no smooth structures. The simplest example in the 4-dimensional E_8 manifold.

Definition 1.15. A function $f : M \to N$ between n-dimensional manifolds M and N is *smooth* at a point $p \in M$ if there is chart (U_M, φ_M) containing p and a chart (U_N, φ_N) containing the image $f(U_M)$ such that $\varphi_N \circ f \circ \varphi_M^{-1} : V_M \to V_N$ is smooth (in the calculus sense).
If f is smooth at every point of M, it is called a *smooth map.*

Definition 1.16. A *(smooth) diffeomorphism* is a smooth bijection whose inverse is also smooth.

Although we showed $\mathbb{R}$ to have distinct smooth structures, these are actually diffeomorphic with the map $x \mapsto x^3$ (note this is smooth because it is relative to the other smooth structure). In fact, $\mathbb{R}^n$ always has a unique smooth structure up to diffeomorphism, except for the case of $\mathbb{R}^4$ which has infinitely many smooth structures.

1.2 Lie Groups

Definition 1.17. A *topological group* is a group $(G, \tau, \circ)$ such that G is a topological space and the maps $\circ : G \times G \to G$ and $^{-1} : G \to G$ are continuous.

Definition 1.18. A *Lie group* is a group $(G, \circ)$ such that G is a smooth manifold and the maps $\circ : G \times G \to G$ and $^{-1} : G \to G$ are smooth.

Example 1.19. $\mathbb{R}^n$ (as one might expect)

Example 1.20. S^1 with modular addition. And, S^3 (identified with unit quaternions (S^1 can also be identified with unit complex numbers))

Interestingly these are the only spheres which are Lie groups (other than the degenerate case of S^0).

Example 1.21. Any torus $T^n = (S^1)^n$ and more generally any product space $M = \prod_{i=1}^{N} M_i$

Example 1.22. $\mathrm{GL}_n(\mathbb{R})$, matrix multiplication is smooth since it is a polynomial in the entries of the matrix

1.3 A Categorical Perspective

Maybe this definition seems arbitrary, i.e. maybe it was be more natural for the maps to be homeomorphisms or diffeomorphisms. Here is some categorical motivation for why this definition is natural.

Some preliminary definitions if necessary:

Definition 1.23. In a category $\mathcal{C}$ a *terminal object* is a object 1 of $\mathcal{C}$ such that for every obejct X there is exactly one morphism $f : X \to 1$.

Think of the trivial group $\{e\}$.

Definition 1.24. In a category $\mathcal{C}$, the *product* of two objects X and Y is an object $X \times Y$ with two morphisms $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ (called the projections) satisfying the following universal property: For any object A and morphisms $f_1 : A \to X$ and $f_2 : A \to Y$ there exists a unique morphism $f : A \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd}
 & A \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y
\end{tikzcd}$$

More succinctly, it is the limit of a diagram of $\{\bullet\ \bullet\}$.

In a category with products, given two morphisms $f : A \to A'$ and $g : B \to B'$ we can define a unique morphism $f \times g : A \times B \to A' \times B'$ using the universal property. Moreover, with the universal property we can define a diagonal morphism $d : G \to G \times G$ by letting $f_1 = f_2 = \mathrm{id}_G$. Products correspond exactly with Cartesian products in **Set**, direct products in **Grp**, direct sums in $\mathbf{Vect}_K$, and product spaces in **Top**.

Definition 1.25. In a category $\mathcal{C}$ with a terminal object 1 and products, a *group object* is an object G together with three morphisms:

- "composition" $m : G \times G \to G$
- "identity" $e : 1 \to G$
- "inverse" $i : G \to G$

Satisfying the following properties:

1. "associativity" $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$
2. "identity" $m \circ (\mathrm{id}_G \times e) = \pi_1$ and $m \circ (e \times \mathrm{id}_G) = \pi_2$ where $\pi_1 : G \times 1 \to G$ and $\pi_2 : 1 \times G \to G$ are the projections.
3. "invertibility" Let $d : G \to G \times G$ be the diagonal morphism and $e_G = e \circ (G \to 1) : G \to G$, then $m \circ (\mathrm{id}_G \times i) \circ d = e_G$ and $m \circ (i \times \mathrm{id}_G) \circ d = e_G$

This lets us define what "groups" are in a variety of different categories:

- In **Set**, group objects correspond to actual groups.
- In **Grp**, group objects correspond to Abelian groups.
- In **Top**, group objects correspond to topological groups.
- In the category of smooth manifolds, group objects correspond to Lie groups.
- In $\mathbf{Vect}_K$, every vector space is a group object in a unique way.
- In the category of algebraic varieties, group objects are called algebraic groups.
- In the category of schemes, group objects are called group schemes.

1.4 Lie Subgroups

A differentiable function is defined exactly like a smooth map except replacing "smooth" with "differentiable".

For a smooth manifold M fix a point p and a chart (U, φ), let $\Gamma(p)$ denote the set of all curves $\gamma : (-1, 1) \to M$ such that $\gamma(0) = p$ and $\varphi \circ \gamma$ is differentiable.

Definition 1.26. The tangent space of M at $p \in M$ is defined to be $T_pM = \Gamma(p)/\sim$ under the equivalence relation $\gamma \sim \psi$ if and only if

$$(\varphi \circ \gamma)'(0) = (\varphi \circ \psi)'(0)$$

It can be shown that this definition is independent of the chart chosen. And T_pM can be made into a vector space by associating $[\gamma]$ to the tangent vector $(\varphi \circ \gamma)'(0)$. As a "sneak peak", when G is a Lie group, the the tangent space at the identity T_eG is isomorphic to its associated Lie algebra.

Definition 1.27. The *differential* of a differentiable function $f : M \to N$ at the point $p \in M$ is the linear map $df_p : T_pM \to T_{f(p)}N$ defined by:

$$df_p([\gamma]) = [f \circ \gamma]$$

Definition 1.28. An *immersion* between to smooth manifolds is a differentiable function whose differential at every point is injective.

Definition 1.29. H is a Lie subgroup of a Lie group G if $H \subseteq G$ and the inclusion map $i : H \hookrightarrow G$ is injective immersion and a group homomorphism.

2 Haar Measure

2.1 Defining the Haar Measure

Remark 2.1. Motivation (From a Representation Theory Perspective) Throughout the representation theory of finite groups, we often want to do "average tricks" where we take a sum over the whole group of some quantity. For example, in the Weyl Unitary Trick, we start with an arbitrary inner product $\langle x, y\rangle'$ on a representation V of G and construct a new inner product such that $\leq g \cdot x, g \cdot y\rangle = \langle x, y\rangle$ by setting $\langle x, y\rangle = \frac{1}{|G|}\sum_{g\in G}\langle g \cdot x, g \cdot y\rangle'$. However, this is not possible for infinite groups (in particular, Lie groups) as the summation could diverge. So, we instead would like to replace the summation with an integration. Moreover, we want the integral to be compatible with the group structure (and with the topological structure if it is a topological group). In particular we want:

$$\int_G f(t)\, dt = \int_G f(gt)\, dt$$

for any $g \in G$. What we can show is that this is possible under certain conditions by constructing a measure on G and using the Lebesgue integral with respect to this measure.

Definition 2.2. A *sigma-algebra* on X is a family of subsets $\Sigma \subseteq \mathcal{P}(X)$ such that:

1. $X \in \Sigma$
2. **Closure under complements** $E \in \Sigma \implies E^c \in \Sigma$ (where X is the universal set)
3. **Closure under countable unions** $E_1, \ldots, E_n \in \Sigma \implies E_1 \cup \cdots \cup E_n \in \Sigma$

Example 2.3. The trivial sigma-algebra $\{\varnothing, X\}$.

Example 2.4. The discrete sigma-algebra $\mathcal{P}(X)$.

Example 2.5. For any given subset $A \subseteq X$, the sigma-algebra $\{\varnothing, A, A^c, X\}$.

Lemma 2.6. *The intersection of sigma-algebras is a sigma-algebra*

Proof. Let Σ_1 and Σ_2 be sigma-algebras on X. $X \in \Sigma_1 \cap \Sigma_2$ since it is contained in both by definition. For any $E \in \Sigma_1 \cap \Sigma_2$, E^c is in both so $E^c \in \Sigma_1 \cap \Sigma_2$. Finally, if $E_1, \ldots, E_n \in \Sigma_1 \cap \Sigma_2$, then $E_1 \cup \cdots \cup E_n$ is in both so $E_1 \cup \cdots \cup E_n \in \Sigma_1 \cap \Sigma_2$. Therefore, $\Sigma_1 \cap \Sigma_2$ is a sigma-algebra. ☐

This allows us to speak of the smallest sigma algebra containing a family of sets $\mathcal{F}$ by taking the intersection of all the sigma algebras containing $\mathcal{F}$ as a subset. For a topological space, a natural sigma-algebra is the smallest one containing all the open sets.

Definition 2.7. The *Borel sigma-algebra* $\mathcal{B}(X)$ of a topological space X is the smallest sigma-algebra that contains all of its open sets.

Definition 2.8. Given a sigma-algebra Σ on a set X, a *measure* is a function $\mu : \Sigma \to \overline{\mathbb{R}}$ such that:

1. **Non-negative** $\forall E \in \Sigma : \ \mu(E) \geq 0$
2. **Null empty set** $\mu(\varnothing) = 0$
3. **Countably additive** For all pairwise disjoint families $\{E_k\}_{k=1}^{\infty} \subseteq \Sigma$ we have:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Example 2.9. The *trivial measure* on any sigma-algebra: $\mu(A) = 0$.

Example 2.10. The *counting measure* on $\mathcal{P}(X)$: $\mu(A) = |A|$ if A is finite and ∞ if A is infinite.

Example 2.11. The *Lebesgue measure*, defined by extending the length function $\ell([a, b]) = b - a$.

On the way to constructing a measure, we might make use of two "stepping stones" that slightly weaken the definition:

Definition 2.12. Given a set X, an *outer measure* is a function $\mu : \mathcal{P}(X) \to \overline{\mathbb{R}}$ such that:

1. **Non-negative** $\forall A \subseteq X : \ \mu(A) \geq 0$
2. **Null empty set** $\mu(\varnothing) = 0$
3. **Monotone** For all $A \subseteq B \subseteq X$, we have: $\mu(A) \leq \mu(B)$
4. **Countably subadditive** For all families $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{P}(X)$ we have:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

Note that even though the name "outer measure" makes it sound like it is a type of measure, not every outer measure is actually a measure. Weakening this definition even more gives:

Definition 2.13. Given a set X and any family of subsets $\mathcal{A}$ (not necessarily a sigma-algebra), a *content* is a function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that:

1. **Non-negative** $\forall A \subseteq \mathcal{A} : \ \mu(A) \geq 0$
2. **Null empty set** $\mu(\varnothing) = 0$
3. **Countably subadditive** For any $A, B \in \mathcal{A}$ such that $A \cap B = \varnothing$ and $A \cup B \in \mathcal{A}$ we have: $\mu(A \cup B) = \mu(A) + \mu(B)$

Now back to measures: For a topological space, we usually want some additional properties:

Definition 2.14. A measure $\mu : \mathcal{B}(X) \to \overline{\mathbb{R}}$ is called *regular* if it satisfies:

1. **Outer regularity** For any Borel set E: $\mu(E) = \inf\{\mu(U) \mid U \text{ open}, E \subseteq U\}$
2. **Inner regularity** For any open set U: $\mu(U) = \sup\{\mu(K) \mid K \text{ compact}, K \subseteq U\}$

For topological group, we would like two additional property:

Definition 2.15. A (left) *Haar measure* on a topological group G is a regular measure which is finite on every compact set and is translation-invariant. That is: for any Borel set E and any $g \in G$, we have: $\mu(gE) = \mu(E)$.

Theorem 2.16 (Haar's Theorem)**.** *Every locally compact Hausdorff topological group has a unique non-trivial Haar measure up to a constant positive multiplicative factor.*

2.2 Constructing the Haar Measure

In what follows, let G be a locally compact Hausdorff topological group. Note this applies to every Lie group. Let $\mathcal{K}$ be the set of all compact subsets of G and $\mathcal{U}$ be the set of all open subsets of G that contain the identity of G.

Remark 2.17. Overview of the construction We are going to construct the Haar measure in a number of stages:

1. First, we will construct a function μ_U on compact sets parameterized by $U \in \mathcal{U}$.
2. Next, we will take a "limit" of these functions to get a function μ defined on compact sets.
3. Then, we will extend μ with inner regularity to a function $\bar{\mu}$ which is defined on all open sets.
4. Then, we will extend $\bar{\mu}$ with outer regularity to a function $\bar{\bar{\mu}}$ which is defined on all subsets.
5. Finally, we will restrict this function to the Borel sets and show that it is the Haar measure.

This is summarized with the following diagram:

$$\begin{tikzcd}
\underset{\text{compact}}{\mu_U} \arrow[r, "\text{``limit''}"] & \underset{\text{compact}}{\mu} \arrow[r, "\text{inner regularity}"] & \underset{\text{open}}{\bar{\mu}} \arrow[r, "\text{outer regularity}"] & \underset{\text{all}}{\bar{\bar{\mu}}} \arrow[r, "\text{restrict}"] & \underset{\text{Borel}}{\eta}
\end{tikzcd}$$

Definition 2.18. For K compact and V such that $V^\circ \neq \varnothing$, define the *covering number* $(K : V)$ to be the smallest number of left translates needed to cover K:

$$(K : V) = \min \left\{ |A| \;\middle|\; K \subseteq \bigcup_{x \in A} xV^\circ \right\}$$

Lemma 2.19. *For any $g \in G$, the map $\varphi_g : G \to G$ defined by $x \mapsto gx$ is a homeomorphism. In particular, for any open set U, we have that gU is also open.*

Proof. Composition is continuous so $V = \{(x, y) \in G^2 \mid xy \in U\}$ is open. Let $(g, G) = \{g\} \times G$ Note that $(g, x) \mapsto x$ is a homeomorphism $(g, G) \to G$. So, since $V \cap (g, G)$ is open in the subspace topology on (g, G), $\pi_2(V \cap (g, G))$ is open in G (where $\pi_2 : (x, y) \mapsto y$ is the projection map). Hence,

$$\pi_2(V \cap (g, G)) = \{x \in G \mid (g, x) \in V\} = \{x \in G \mid gx \in U\} = \varphi_g^{-1}(U)$$

Hence, φ_g is continuous. Clearly $\varphi_g^{-1} = \varphi_{g^{-1}}$ so φ_g is bijective. And since g was arbitrary, φ_g^{-1} is also continuous. Therefore, φ_g is a homeomorphism. □

Proposition 2.20. *$(K : V)$ always exists and is always a non-negative integer. Moreover, we have $(K : V) = 0$ if and only if $K = \varnothing$.*

Proof. For any set, letting $A = G$ will most certainly cover K (as well as the whole space) with set which are open by the above lemma since V° is open. Since K is compact, there is a finite subcover. Hence, $(K : V)$ is finite. Moreover, since it is the minimum of a set of non-negative integers, it is a non-negative integer. If $K = \varnothing$ then $A = \varnothing$ covers it so $(K : V) = 0$. If $(K : V) = 0$ then $A = \varnothing$ covers K, meaning K is empty. □

Fix a compact set with non-empty interior K_0, this must exist since G is locally compact. For any $U \in \mathcal{U}$, define $\mu_U : \mathcal{K} \to \mathbb{R}$ by:

$$\mu_U(K) = \frac{(K : U)}{(K_0 : U)}$$

Lemma 2.21. *For each U, we have: $\mu_U(K) \leq (K : K_0)$ for all $K \in \mathcal{K}$*

Proof. This equivalent to $(K : U) \leq (K : K_0)(K_0 : U)$. K_0 is covered by $n = (K_0 : U)$ translates of U, say by $\{g_i\}_{i=1}^{n}$. Similarly, K is covered by $m = (K : K_0)$ translates of K_0, say by $\{h_j\}_{j=1}^{m}$. Thus, K is covered by the translates of U by $\{g_i h_j\}_{i,j=1,1}^{n,m}$. Hence, $(K : U)$ is at most mn. Therefore,

$$(K : U) \leq mn = (K : K_0)(K_0 : U)$$

□

Define the space $X = \prod_{K \in \mathcal{K}} [0, (K : K_0)]$. By the lemma, $\mu_U(K) \leq (K : K_0)$, so we can think of each μ_U as a point in X, i.e. $(\mu_U(K_1), \mu_U(K_2), \dots)$.

Remark 2.22. The Intuitive Idea We want to measure the relative size of K to K_0, but for each μ_U all we can use to measure it is U. Each U is a like a yardstick without any markings, all we can do is lay it down and see how many it takes to cover K and K_0 and take their quotient. A very large U will give a very imprecise measurement, like "roughly a $3 : 4$ ratio". If we want a more precise measurement, we need to use a smaller U to get something like a "$29 : 37$ ratio". The hope is that as we take smaller and smaller U's this ratio will converge to a single limit. In the picture, this should look like a ball shrinking in on a point with each concentric ring corresponding to smaller and smaller open sets.

Define the following set for each $V \in \mathcal{U}$:

$$C(V) = \overline{\{\mu_U \mid U \in \mathcal{U}, U \subseteq V\}}$$

Lemma 2.23. *$\{C(V)\}_{V \in \mathcal{U}}$ has the finite intersection property (the intersection of finitely many sets in non-empty)*

Proof. Since each V_k is open, $\bigcap_{k=1}^n V_k$ will also be open (and will also contain the identity). As such, $\mu_{\bigcap_{k=1}^n V_k}$ is defined and is contain in each $C(V_k)$. Thus, $\bigcap_{k=1}^n C(V_k)$ is non-empty. ☐

We now use two classical results from topology: Tychnoff's Theorem (A product of compact spaces is compact) and that a space is compact if and only if any family of closed sets with the finite intersection property has a non-empty intersection. By Tychonoff's Theorem, X is compact. And, by the classification of compact spaces $\bigcap_{V \in \mathcal{U}} C(V)$ is non-empty. Thus, we can pick an arbitrary function in $\bigcap_{V \in \mathcal{U}} C(V)$. Call this function μ.

Proposition 2.24. *$\mu(K_1) \leq \mu(K_2)$ whenever $K_1 \subseteq K_2$.*

Proof. Clearly $\mu_U(K_1) \leq \mu_U(K_2)$ for all U since every covering of K_2 is also a covering of K_1. Now, thinking of $f \in X$, as a function, evaluating $f \mapsto f(K)$ can be though of as a projection map. Thus, $h : f \mapsto f(K_2) - f(K_1)$ is continuous as a function $X \to \mathbb{R}$. Thus, since $\mu_U(K_1) \leq \mu_U(K_2)$ for all U, h is also non-negative on $C(V)$ (because it is continuous). So, $h(\mu) \geq 0$ meaning $\mu(K_1) \leq \mu(K_2)$. ☐

Lemma 2.25. *Let K be compact and U be open with $K \subseteq U$. Then, there exists some $V \in \mathcal{U}$ such that $KV \subseteq U$.*

Proof. For any $x \in K$, let $W_x = x^{-1}U$. Since $x \in U$ we have $x^{-1}x = e \in W_x$ so $W_x \in \mathcal{U}$. Since multiplication is continuous, we can find a set $V_x \in \mathcal{U}$ such that $V_x V_x \subseteq W_x$ (take preimage and intersect the projections onto G, this is open since it is intersection of open sets and contains the identity). Then, $\{xV_x \mid x \in K\}$ is an open cover of K so we can find $x_1, \ldots, x_n$ such that $\{x_i V_{x_i}\}_{i=1}^n$ covers K. Define $V = \bigcap_{i=1}^n V_{x_i}$. Now for any $k \in K$ there is some x_k such that $k \in x_k V_{x_k}$. Thus,

$$kV \subseteq x_k V_{x_k} V_{x_k} \subseteq x_k W_{x_k} = U$$

Therefore, $KV \subseteq U$. ☐

Proposition 2.26. *μ is a content on compact sets (that is a measure but with the countable union property replaced with finite unions).*

Proof. Clearly, μ is non-negative since $\mu \in X$ and $\mu(\varnothing) = 0$ since $(\varnothing : U) = 0$ for all U. Now, we show μ is finitely additive in several steps.

First: that $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$. Clearly $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$ for each U since every union of covering of K_1 and K_2 is a covering of $K_1 \cup K_2$. Just as in the previous lemma, the map $h : f \mapsto f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is continuous as a function $X \to \mathbb{R}$. And, since $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$ for each U, h is non-negative for each $C(V)$. Thus, since $h(\mu) \geq 0$ we have $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$.

Second: that $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ if $K_1U^{-1} \cap K_2U^{-1} = \varnothing$. Let $\{g_i\}_{i=1}^{(K_1 \cup K_2 : U)}$ be representatives that cover $K_1 \cup K_2$. Suppose for contradiction that for some g_i, we have that g_iU intersects both K_1 and K_2. Then, $g_i \in K_1U^{-1} \cap K_2U^{-1}$. A contradiction. Thus, each g_iU intersects exactly one of K_1 and K_2. Thus, by taking two subsequences we can find two covers of K_1 and K_2 respectively. Hence, $\mu_U(K_1) + \mu_U(K_2) \leq \mu_U(K_1 \cup K_2)$. Combining this with the previous gives the result.

Third: that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$ if $K_1 \cap K_2 = \varnothing$. Since G is Hausdorff, we can find open sets $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$ such that $U_1 \cap U_2 = \varnothing$. By the above lemma, there exists $V_1, V_2 \in \mathcal{U}_1$ such that $K_1V_1 \subseteq U_2$ and $K_2V_2 \subseteq U$. Let $V = V_1 \cap V_2$. Then $K_1V \cap K_2V = \varnothing$ since U_1 and U_2 are disjoint. Thus, for any $U \in \mathcal{U}$ with $U \subseteq V^{-1}$, we have $K_1U^{-1} \cap K_2U^{-1} = \varnothing$. So, by the previous step $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$. Hence, the map $h(f) = 0$ for all $f \in C(V^{-1})$. In particular, $h(\mu) = 0$ so $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$.

Therefore, μ is a content on compact sets. □

Now, μ is only defined on compact sets. In order to extend it to every Borel set, we will first extend with inner regularity to open sets and then with outer regularity to Borel sets. First we define the function $\bar{\mu}$ on open sets by:

$$\bar{\mu}(U) = \sup\{\mu(K) \mid K \in \mathcal{K}, K \subseteq U\}$$

For any set K' which is both open and compact, since $\mu(K') \in \{\mu(K) \mid K \in \mathcal{K}, K \subseteq K'\}$ we have that $\mu(K') \leq \bar{\mu}(K')$ and since we have $\mu(K) \leq \mu(K')$ for any $K \subseteq K'$ we know $\bar{\mu}(K') \leq \mu(K')$. Thus, $\mu(K') = \bar{\mu}(K')$ so $\bar{\mu}$ agrees with μ when their domains overlap. Furthermore, if $U_1 \subseteq U_2$ we still have $\bar{\mu}(U_1) \leq \bar{\mu}(U_2)$, because $\{\mu(K) \mid K \in \mathcal{K}, K \subseteq U_1\} \subseteq \{\mu(K) \mid K \in \mathcal{K}, K \subseteq U_2\}$.

Now, we define the function $\bar{\bar{\mu}}$ on the power set of G by:

$$\bar{\bar{\mu}}(A) = \inf\{\bar{\mu}(U) \mid U \in \tau, A \subseteq U\}$$

For any open set U', since $\bar{\mu}(U') \in \{\bar{\mu}(U) \mid U \in \tau, U' \subseteq U\}$ we have that $\bar{\bar{\mu}}(U') \leq \bar{\mu}(U')$ and since we have $\bar{\mu}(U') \leq \bar{\mu}(U)$ for any $U' \subseteq U$ we know $\bar{\mu}(U') \leq \bar{\bar{\mu}}(U')$. Thus, $\bar{\mu}(U') = \bar{\bar{\mu}}(U')$ so $\bar{\bar{\mu}}$ agrees with $\bar{\mu}$ on open sets. Similarly, for a compact set K', every open set $U \supseteq K'$ contains K' trivially as a subset so $\bar{\mu}(U) \geq \mu(K')$. Thus, $\bar{\bar{\mu}}(K') \geq \mu(K')$. Furthermore, if $A_1 \subseteq A_2$ we still have $\bar{\bar{\mu}}(A_1) \leq \bar{\bar{\mu}}(A_2)$, because $\{\bar{\mu}(U) \mid U \in \tau, A_2 \subseteq U\} \subseteq \{\bar{\mu}(U) \mid U \in \tau, A_1 \subseteq U\}$.

Proposition 2.27. *$\bar{\bar{\mu}}$ is regular.*

Proof. For a Borel set E, since $\bar{\bar{\mu}}$ agrees with $\bar{\mu}$ on open sets:

$$\bar{\bar{\mu}}(E) = \inf\{\bar{\mu}(U) \mid U \in \tau, E \subseteq U\} = \bar{\bar{\mu}}(A) = \inf\{\bar{\bar{\mu}}(U) \mid U \in \tau, E \subseteq U\}$$

Hence, we have outer regularity. For an open set U, we have $\bar{\bar{\mu}}(U) \geq \sup\{\bar{\bar{\mu}}(K) \mid K \in \mathcal{K}, K \subseteq U\}$ by monotonicity. And, since $\bar{\bar{\mu}}$ agrees with $\bar{\mu}$ on open sets:

$$\bar{\bar{\mu}}(U) = \bar{\mu}(U) = \sup\{\mu(K) \mid K \in \mathcal{K}, K \subseteq U\} \leq \sup\{\bar{\bar{\mu}}(K) \mid K \in \mathcal{K}, K \subseteq U\}$$

where the last inequality is by $\bar{\bar{\mu}}(K) \geq \mu(K)$ for a compact set K. Hence, we have inner regularity. ☐

Restricting $\bar{\bar{\mu}}$ to the Borel sets gives us the Haar measure!

Lemma 2.28. *If $K \in \mathcal{K}$ and $K \subseteq U_1 \cup U_2$ for open sets U_1 and U_2, then there are compact sets K_1 and K_2 such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$ and $K = K_1 \cup K_2$.*

Proof. Let $L_1 = K \setminus U_1$ and $L_2 = K \setminus U_2$. Because G is Hausdorff, K is closed, so L_1 and L_2 are both closed. Since they are closed subsets of a compact set L_1 and L_2 are also compact. Since $K \subseteq U_1 \cup U_2$, $L_1 \cap L_2 = \varnothing$. Thus, since G is Hausdorff, L_1 and L_2 can be separated by disjoint open sets, V_1 and V_2. Let $K_1 = K \setminus V_1$ and $K_2 = K \setminus V_2$. Similarly to L_1 and L_2, we have that K_1 and K_2 are compact. And for $i \in \{1, 2\}$, we have:

$$K_i = K \setminus V_i \subseteq K \setminus L_i = K \setminus (K \setminus U_i) = K \cap U_i \subseteq U_i$$

And, $K_1 \cup K_2 = (K \setminus V_1) \cup (K \setminus V_2) = K \setminus (V_1 \cup V_2) = K$ since V_1 and V_2 are disjoint. ☐

Proposition 2.29. *$\bar{\bar{\mu}}$ is an outer measure on G.*

Proof. Clearly we still have $\bar{\bar{\mu}}(\varnothing) = 0$ since $\bar{\bar{\mu}}$ agrees with μ on compact sets. And $\bar{\bar{\mu}}$ is non-negative because supremums and infimums of non-negative numbers are still non-negative.

For countable subadditivity, we will first prove it for open sets. Let $\{U_n\}_{n=1}^{\infty} \subseteq \tau$. For any compact subset $K \subseteq \bigcup_{n=1}^{\infty} U_n$ there is some $N \in \mathbb{N}$ such that $K \subseteq \bigcup_{n=1}^{N} U_n$. By applying the above lemma inductively, we can find compact sets $K_1, \ldots, K_N$ such that $K = \bigcup_{n=1}^{N} K_n$ and $K_n \subseteq U_n$ for each $1 \leq n \leq N$. Then, applying $\mu(K) \leq \mu(K_1) + \mu(K_2)$ for $K = K_1 \cup K_2$ inductively (since $\bar{\bar{\mu}}$ agrees with μ on compact sets), we have:

$$\bar{\bar{\mu}}(K) \leq \sum_{n=1}^{N} \bar{\bar{\mu}}(K_n) \leq \sum_{n=1}^{N} \bar{\bar{\mu}}(U_n) \leq \sum_{n=1}^{\infty} \bar{\bar{\mu}}(U_n)$$

Hence, it is true for open sets, since $\bar{\bar{\mu}}$ agrees with $\bar{\mu}$ on open sets and μ on compact sets:

$$\bar{\bar{\mu}}\left(\bigcup_{n=1}^{\infty} U_n\right) = \sup\left\{\bar{\bar{\mu}}(K) \,\middle|\, K \subseteq \bigcup_{n=1}^{\infty} U_n, K \in \mathcal{K}\right\} \leq \sum_{n=1}^{\infty} \bar{\bar{\mu}}(U_n)$$

Now, we prove countable subadditivity for an arbitrary family $\{A_n\}_{n=1}^{\infty}$. If $\sum_{n=1}^{\infty} \bar{\bar{\mu}}(A_n) = \infty$ then the inequality is trivial, so we can assume $\sum_{n=1}^{\infty} \bar{\bar{\mu}}(A_n)$ is finite. Fix $\varepsilon > 0$. Since $\bar{\bar{\mu}}$ agrees with $\bar{\mu}$ on open sets, by the infimum we can find for each A_n an open set U_n such that $A_n \subseteq U_n$ and $\bar{\bar{\mu}}(U_n) \leq \bar{\bar{\mu}}(A_n) + \varepsilon/2^n$. Then, we have:

$$\bar{\bar{\mu}}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \bar{\bar{\mu}}\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \bar{\bar{\mu}}(U_n) \leq \sum_{n=1}^{\infty} \bar{\bar{\mu}}(A_n) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \bar{\bar{\mu}}(A_n) + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we have:

$$\bar{\bar{\mu}}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \bar{\bar{\mu}}(A_n)$$

Therefore, $\bar{\bar{\mu}}$ is an outer measure. □

Now define $\eta = \bar{\bar{\mu}}|_{\mathcal{B}(G)}$, the restriction of $\bar{\bar{\mu}}$ to the Borel sets. We now invoke a classical result of measure theory: Caratheodory's Criterion. It states that for an outer measure μ^*, the sets E satisfying $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all $A \subseteq G$ form a sigma-algebra. Such a set E is called Caratheodory measurable. In general, this can be used to extract a measure from any outer measure. But, we specifically want a measure on the Borel sigma-algebra. So, all we need to do is check that each Borel set is Caratheodory measurable.

Proposition 2.30. *$\eta : \mathcal{B}(G) \to \mathbb{R}$ is a measure.*

Proof. To show that each Borel set is Caratheodory measurable, it suffices to show that each open set is Caratheodory measurable because any sigma-algebra containing a family of set will contain the sigma-algebra it generates. So, let $U \subseteq G$ be open and let $A \subseteq G$. If $\bar{\bar{\mu}}(A) = \infty$ then trivially $\bar{\bar{\mu}}(A) \geq \bar{\bar{\mu}}(A \cap U) + \bar{\bar{\mu}}(A \cap U^c)$. If $\bar{\bar{\mu}}(A) < \infty$, fix $\varepsilon > 0$. Since $\bar{\bar{\mu}}(A) = \inf\{\bar{\mu}(U) \mid U \in \tau, A \subseteq U\}$, we can find an open set V such that $A \subseteq V$ and $\bar{\bar{\mu}}(V) \leq \bar{\bar{\mu}}(A) + \varepsilon$. And since on open sets $\bar{\mu}(U)$ is the supremum of $\{\mu(K) \mid K \in \mathcal{K}, K \subseteq U\}$ we can find compact $K \subseteq V \cap U$ such that $\bar{\bar{\mu}}(V \cap U) - \varepsilon \leq \bar{\bar{\mu}}(K)$. Similarly, we can find compact $L \subseteq V \cap K^c$ such that $\bar{\bar{\mu}}(V \cap K^c) - \varepsilon \leq \bar{\bar{\mu}}(L)$. Since $K \subseteq U$ we have $V \cap U^c \subseteq V \cap K^c$, so:

$$\bar{\bar{\mu}}(V \cap U^c) - \varepsilon \leq \bar{\bar{\mu}}(V \cap K^c) - \varepsilon \leq \bar{\bar{\mu}}(L)$$

Now, we also have $A \cap U \subseteq V \cap A$ and $A \cap U^c \subseteq V \cap U^c$ and $\bar{\bar{\mu}}$ retains the property of being a content on compact sets so:

$$\begin{aligned}
\bar{\bar{\mu}}(A \cap U) + \bar{\bar{\mu}}(A \cap U^c) - 2\varepsilon &\leq \bar{\bar{\mu}}(V \cap U) - \varepsilon + \bar{\bar{\mu}}(V \cap U^c) - \varepsilon \\
&\leq \bar{\bar{\mu}}(K) + \bar{\bar{\mu}}(L) && \text{Definitions of } K \text{ and } L \\
&= \bar{\bar{\mu}}(K \cup L) && \text{Content on compact sets} \\
&\leq \bar{\bar{\mu}}((V \cap U) \cup (V \cap K^c)) && \text{Definitions of } K \text{ and } L \\
&\leq \bar{\bar{\mu}}(V) && \text{Since } (V \cap U) \cup (V \cap K^c) \subseteq V \\
&\leq \bar{\bar{\mu}}(A) + \varepsilon && \text{Definition of } V
\end{aligned}$$

Hence, $\bar{\bar{\mu}}(A \cap U) + \bar{\bar{\mu}}(A \cap U^c) - 3\varepsilon \leq \bar{\bar{\mu}}(A)$ and since ε was arbitrary, we have:

$$\bar{\bar{\mu}}(A \cap U) + \bar{\bar{\mu}}(A \cap U^c) \leq \bar{\bar{\mu}}(A)$$

Therefore, U is Caratheodory measurable and η restricted to the Borel sets is a measure. ☐

Now, we finally come to the easy part. After all of this setup, we can prove that η is indeed a Haar measure.

Proposition 2.31. *η is non-trivial.*

Proof. $\mu_U(K_0) = 1$ for all U and the map $f \mapsto f(K_0)$ is continuous as $X \to \mathbb{R}$ so $\mu(K_0) = 1$ and since η agrees with μ on compact sets, $\eta(K_0) = 1$. Therefore, η is non-trivial. ☐

Theorem 2.32. *$\eta : \mathcal{B}(G) \to \mathbb{R}$ is a Haar measure on G.*

Proof. Since $\mu \in X$ and η agrees with μ on compact sets, η is finite on compact sets.

Since $\bar{\bar{\mu}}$ is regular and η is a restriction of $\bar{\bar{\mu}}$, it is also regular.

For translation-invariance, it suffices to show that η is translation invariant on compact sets, because the supremums and infimums of equal sets are also equal. Fix $g \in G$ and $K \in \mathcal{K}$. Translations by $x_1, \ldots, x_n$ cover K if and only if translations by $gx_1, \ldots, gx_n$ cover gK so $(K : U) = (gK : U)$ for any $U \in \mathcal{U}$. Hence, $\mu_U(K) = \mu_U(gK)$. Thus, the continuous map $X \to \mathbb{R}$ given by $f \mapsto f(K) - f(gK)$ is 0 on every $C(U)$. So, $\mu(K) = \mu(gK)$. Since η agrees with μ on compact sets, it follows that $\eta(K) = \eta(gK)$. Hence, η is translation-invariant.

Therefore, η is a Haar measure. ☐