05/24/22 Notes

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- Next meeting time: 10am on Thursday at Noyes 1

1 Schur's Lemma

Theorem 1.1. *Let $f_1 : G \to GL(V_1)$ and $f_2 : G \to GL(V_2)$ be two irreducible representations of G, let f be linear map of V_1 into V_2 such that it intertwines with two representations, i.e. $f_2(s) \circ f = f \circ f_1(s)$, $\forall s \in G$, then*

(i) If f_1 and f_2 are not isomorphic, then f=0

(ii) If f_1 and f_2 are isomorphic, there exists basis such that $f = \lambda I$, a homothety.

Proof. (i) If f_1 and f_2 are not isomorphic, note that the ker(f) is G-stable. Since we have, $\forall v \in ker(f)$, $\forall s \in G$, $f(f_1(s)v) = f_2(s)f(v) = 0$, thus $f_1(s)v \in ker(f)$. Suppose $f \neq 0$, then since $ker(f)$ is a G-module and V_1 is irreducible, we must have $ker(f) = 0$. Then $f(V_1)$ is a isomorphic copy of V_1 in V_2. Then, $V_1 \cong_G f(V_1) \cong_G V_2$. So suppose we are in the second case,

(ii) View $V_1 = V_2$ and $f_1 = f_2$, we see the eigenspace of f is also G-stable. Suppose $v \in V_\lambda$, $\forall s \in G$, $f(f_1(s)v) = f_1(s)f(v) = \lambda f_1(s)v$, thus $f_1(s)v \in V_\lambda$. Since we are in an algebraically closed field, we must have one eigenvalue λ whose eigenspace V_λ is nontrivial. Then, since it is G-stable and V_1 is irreducible, $V_\lambda = V_1$. Thus, $f = \lambda I$.

□

2 Class Functions and Characters

Definition 2.1. For a group G and complex numbers $\mathbb{C}$, Class(G) is the set of all functions $f : G \to \mathbb{C}$ such that $\forall t, s \in G$, $f(tst^{-1}) = f(s)$.

Lemma 2.2. *Class(G) is a inner product vector space with pointwise addition and scalar multiplication.*

Proof. Since functions $f : G \to \mathbb{C}$ is a vector space over $\mathbb{C}$, it suffices to verify that Class(G) is closed under operations. For $\alpha, \beta \in \mathbb{C}$, for $f, g \in Class(G)$,$\forall t, s \in G$,

$$\alpha f(tst^{-1}) + \beta g(tst^{-1}) = \alpha f(s) + \beta g(s)$$

Thus, $\alpha f(s)+\beta g(s) \in Class(G)$. Class(G) is a vector space.

For the inner product, define

$$(f|g) := \frac{1}{|G|}\sum_{t\in G} f(t)\overline{g(t)}$$

This is the standard inner product for complex vector space but with a factor $\frac{1}{|G|}$, which makes it still a inner product. □

Theorem 2.3 (Serre)**.**

(i) If χ is a character of irreducible representation, then $(\chi|\chi)$ = 1

(ii) If χ and $\chi^{'}$ are character of non-isomorphic irreducible representations, then $(\chi|\chi^{'}) = 0$

The proof relies on the Schur's lemma and a nice averaging trick to give relations between the entries of the matrix, and introduces a new symmetric bilinear form that is easy to work with and is the same as the inner product when apply to character.

Lemma 2.4. *For a linear map h from V_1 to V_2 and respectively, their representations ϕ_1 and ϕ_2, define*

$$h^{'} = \frac{1}{g}\sum_{t\in G}\left(f_2(t)\right)^{-1} h f_1(t)$$

Then,

(i) If f_1 and f_2 are not isomorphic, we have $h^{'} = 0$

(ii) If they are isomorphic, we have basis such that $h^{'}$ is a homothety of ratio $(1/n)\operatorname{Tr}(h)$, where n is the dimension of $V_1 \cong_G V_2$.

Proof. The averaging gives us a intertwiner $h^{'}$, i.e. $f_2(t)^{-1}h^{'}f_1(t) = h^{'}$, $\forall t \in G$. Thus, (i) is clear from Schur's lemma. For (ii), we need to check the number on the diagonal. By a straightforward computation from the definition, we have

$$\operatorname{Tr}(h^{'}) = \frac{1}{|G|}\sum_{t\in G}\operatorname{Tr}((f_1(t))^{-1} h f_1(t)) = \operatorname{Tr}(h)$$

Thus, since for $h^{'} = \lambda I$, $\operatorname{Tr}(h^{'}) = n\lambda = \operatorname{Tr}(h)$, $h^{'} = (1/n)Tr(h) * I$. □

Definition 2.5. Define a symmetric bilinear form for scalar functions on G

$$\langle\phi,\psi\rangle = \sum_{t\in G}\phi(t)\psi\left(t^{-1}\right)$$

.

It is worth noting that when we plug in characters, the form is the same as the inner product.

Now, we rewrite $h^{'}$, f_1 and f_2 elementwise $f_1(t) = (r_{j_1 i_1}(t))$, $f_2(t) = (r_{i_2 j_2}(t))$, $h = (h_{ij})$, and $h' = (h^{'}_{ij})$. A direct computations gives us the relation,

$$h^{'}_{i_2 i_1} = \frac{1}{|G|}\sum_{t,j_1,j_2} r_{i_2 j_2}\left(t^{-1}\right) h_{j_2 j_1} r_{j_1 i_1}(t) \tag{1}$$

$$= \frac{1}{|G|}\sum_{j_1,j_2}(\sum_{t} r_{i_2 j_2}\left(t^{-1}\right) r_{j_1 i_1}(t)) h_{j_2 j_1} \tag{2}$$

$$= \sum_{j_1,j_2}\langle r_{j_1 i_1}, r_{i_2 j_2}\rangle h_{j_2 j_1} \tag{3}$$

Corollary 2.6. *In case (i), we have*

$$\langle r_{j_1 i_1}, r_{i_2 j_2} \rangle = 0$$

for any i_1, i_2, j_1, j_2.

Proof. Observe that (3) is a linear form with respect to $h_{j_2 j_1}$ and vanishes for all $h_{j_2 j_1}$. Thus, the coefficients are 0. □

Corollary 2.7. *In case (ii), we have*

$$\langle r_{j_1 i_1}, r_{i_2 j_2} \rangle = \frac{1}{n}\delta_{i_2 i_1}\delta_{j_2 j_1} = \begin{cases} 1/n & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

for any i_1, i_2, j_1, j_2.

Proof. We split the argument into two cases:

1. $i_1 \neq i_2$: we have (3) to be a zero linear form, thus the coefficients are 0.
2. $i_1 = i_2$: Note that, in this case we have

$$h'_{ii} = \frac{1}{n}\,\mathrm{Tr}(h) = \frac{1}{|G|}\sum_{j_1, j_2} \frac{1}{n}\delta_{j_1 j_2} h_{j_2, j_1}$$

   Thus, the coefficients of this linear form and the coefficients of (3) must be equal since they are equal for all inputs.

□

Proof of Orthogonality Theorem 2.3. We have

$$\begin{aligned}(\chi|\chi') &= \langle \chi, \chi' \rangle \\ &= \langle \mathrm{Tr}(f), \mathrm{Tr}(f') \rangle \\ &= \langle \sum_{i=1}^{n} r_{ii}, \sum_{j=1}^{n} r'_{jj} \rangle\end{aligned}$$

If χ and χ' are non-isomorphic, by Corollary 2.6, all $\langle r_{ii}, r'_{jj} \rangle = 0$. Thus, $(\chi|\chi') = 0$

If χ and χ' are isomorphic, by Corollary 2.7, $(\chi|\chi') = \langle \sum_{i=1}^{n} r_{ii}, \sum_{j=1}^{n} r_{jj} \rangle = \sum_{i=1}^{n} \langle r_{ii}, r_{ii} \rangle = n * \frac{1}{n} = 1$ □

3 Consequences of Orthogonality of Characters

Theorem 3.1. *For a linear representation V with character ϕ, suppose V decomposes into irreducible representations,*

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k.$$

Then, if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to $(\phi|\chi)$.

Proof. Suppose W_i has character χ_i, we know that $\phi = \chi_1 + \ldots + \chi_k$. Thus, $(\phi|\chi) = \sum_{i=1}^{k}(\chi_i|\chi) = \sum_{i:W_i \text{ isomorphic to } W}(\chi_i|\chi) = \sum_{i:W_i \text{ isomorphic to } W} 1 = \#$ of W_i that is isomorphic to W □

Corollary 3.2. *The number of W_i isomorphic to W does not depend on the decomposition.*

Since $(\phi|\chi)$ does not depend on the decomposition.

Corollary 3.3. *Two representations with the same character are isomorphic.*

We use every irreducible representations to "test" the number of copies of it appeared in two representations.Since they have the same character, we get two equal decompositions up to isomorphism.

Theorem 3.4. *If χ is a character of representation V, $(\chi|\chi)$ is a positive integer and if $(\chi|\chi) = 1$, V is irreducible.*

Proof. First, we group together the isomorphic components in the decomposition, that is $V = \oplus_{i=1}^{l} n_i W_i$ and $W_i \ncong_G W_j$ if $i \neq j$. Easy to see n_i is a positive integer. Then, $(\phi|\phi) = \sum_{i=1}^{l} n_i^2 \in \mathbb{Z}^+$. Also, if $(\phi|\phi) = 1$, we must have one $n_i = 1$ and 0 for all other n_j. Thus, V is irreducible. □

4 Characters Tables

A character table is a compact description of the irreducible characters of a group. The columns are the conjugacy classes of the group, and the rows are the irreducible characters.

Example 4.1. Expanded Character table for S_3

	(1)	(12)	(23)	(13)	(123)	(132)
trivial	1	1	1	1	1	1
sign	1	-1	-1	-1	1	1
standard	2	0	0	0	-1	-1

Definition 4.2. The standard representation of S_n is a subrepresentation of the permutation representation. Let V be a n dimensional vector space, and $\beta = \{v_1, \ldots, v_n\}$ be a basis. Then S_n acts naturally on V by permuting the basis vectors. The $n-1$ dimensional subspace of V, $\{(x_1, \ldots, x_n) | \sum_{i=1}^{n} x_i = 0\}$ is the standard representation.

How does one compute such a character table? Say we only knew two of the irreducible representations, how could we find the character values of the third?

	(1)	(12)	(23)	(13)	(123)	(132)
trivial	1	1	1	1	1	1
sign	1	-1	-1	-1	1	1
p	a	b	b	b	c	c

Lets call this mystery representation p with character χ. $a = \chi(1) = 2$ can be deduced from first two well-known representations by the fact that $\sum_{i=1}^{3} deg(W_i)^2 = |G| = 6$ deduced from regular representation. Then using the fact that irreducible characters are orthogonal, we can produce a system of linear equations

$$\overline{a} - 3\overline{b} + 2\overline{c} = 0$$

$$\overline{a} + 3\overline{b} + 2\overline{c} = 0$$

This implies $b = 0$ and $c = -1$ We can now compress this table by aggregating members of the same conjugacy class. For S_n it is well known that the conjugacy classes are cycles with same cycle type.

	(1)	(12)	(123)
trivial	1	1	1
sign	1	-1	1
standard	2	0	-1

To calculate character tables of general finite groups, there is the Burnside-Dixon-Schneider algorithm which reduces the problem to computing eigenvalues. For the symmetric group, there is the Murnaghan-Nakayama Rule.
There are a plethora of open problems, even about the character table of S_n.

Fact 1. *The sum across any row of a compressed character table of a finite group is a non negative integer.*

Proof. Let f be a representation of G acting on itself by conjugation. i.e $g \cdot h = ghg^{-1}$. It is well known that the multiplicity of χ in χ_f is $(\chi_f|\chi) = \sum_K \chi(K)$, where K is a conjugacy class of G (Exercise 7.71, [4]), but that is simply the row sum of χ. □

The problem is to now give a combinatorial interpretation of the row sums, re-explaining why they are non negative (Problem 11, [5]). For certain rows [1] gives an explanation.

Fact 2. *The density of* 0 *in the extended character table of* S_n *tends towards 1 as* $n \to \infty$,.

This is shown in [2].

References

[1] Kirby Baker & Edward Early (2016) *Character Polynomials and Row Sums of the Symmetric Group*, http://edwarde.create.stedwards.edu/rowsums.pdf

[2] Alexander R. Miller (2013) *The probability that a character value is zero for the symmetric group*, https://arxiv.org/abs/1306.1219

[3] Jeane P. Serre (1977) *Linear representations of finite groups.* Springer-Verlag

[4] Richard P. Stanley & Sergey Fomin (1999) *Enumerative Combinatorics, vol 2.* Cambridge University Press.

[5] Richard P. Stanley (1999) *Positivity Problems and Conjectures in Algebraic Combinatorics.* Massachusetts Institute of Technology