06/08/22 Notes

Casey Appleton, Judy Chiang and Dinglong Wang

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These notes are based on a talk given by Prof. Alex Woo.

1 "Abstract Nonsense"

We begin by making some definitions and stating some general results that will become useful later on. In all of the following, let V, W be representations of a group G over a field $\mathbb{K}$

Definition 1.1. $[V, W]_{\mathbb{K}}$ is the $\mathbb{K}[G]$-module of $\mathbb{K}$-linear maps $f : V \to W$ with $(g \cdot f)(v) := g \cdot f(g^{-1} \cdot v) \forall v \in V \forall g \in G$.

Definition 1.2. $V \otimes W$ is the $\mathbb{K}[G]$-module $V \otimes_{\mathbb{K}} W$ with G acting diagonally: $g \cdot (v \otimes w) := g \cdot v \otimes g \cdot w$. (Note that iterating this to define products of any finite number of $\mathbb{K}[G]$-modules can be easily seen to be equivalent to the diagonal action of G on the tensor product of all of them.)

Definition 1.3. $V^G := \{v \in V : \forall g \in G, g \cdot v = v\}$ as a $\mathbb{K}$-vector space.

Definition 1.4. Let X be a $\mathbb{K}$-vector space. Then $X_{triv} := X$, with the trivial action of G: $\forall g \in G, g \cdot (-) := id_X$. When no action of G on X has been defined, X_{triv} may also just be denoted by X.

Definition 1.5. $V^* := [V, \mathbb{K}_{triv}]_{\mathbb{K}}$

Remark 1.6. Recall the Categories K-Vect, whose objects are vector spaces with morphisms being linear maps between them, and $\mathbb{K}[G]$-Mod, whose objects are $\mathbb{K}[G]$-modules with module homomorphisms (G-equivariant maps) between them. In this way, many of the constructions defined above become functors. $[-,-]_{\mathbb{K}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, $(- \otimes -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(-)^G : \mathcal{C} \to \mathcal{D}$, and $(-)_{triv} : \mathcal{D} \to \mathcal{C}$ are all functors, where $\mathcal{C}$ is $\mathbb{K}[G]$-Mod, and $\mathcal{D}$ is $\mathbb{K}$-Vect. Additionally, essentially by definition, $\mathrm{Hom}_{\mathbb{K}[G]}(-,-) := \mathrm{Hom}_{\mathcal{C}}(-,-) = ([-,-]_{\mathbb{K}})^G$ as vector spaces.

Lemma 1.7. *$(-)^G$ and $(-)_{triv}$ are right and left adjoint functors respectively*

Proof. The correspondence in one direction is given by sending a G-equivariant map $A_{triv} \to B$ to the map between the underlying vector spaces, which can be seen to have image lying in B^G. In the other direction, a map $A \to B^G$ gets sent to $A_{triv} \to (B^G)_{triv}$ using the fact the $(-)_{triv}$ is a functor, and then this is followed by the inclusion $(B^G)_{triv} \hookrightarrow B$, (Since $(B^G)_{triv}$ is a subrepresentation of B). The rest of the facts needed for this claim can be easily and methodically verified. $\square$

We find that for the internal-hom and tensor product, we get something stronger than just an adjunction.

Proposition 1.8. *There is a natural isomorphism* $\Phi : [-_1 \otimes_{\mathbb{K}} -_2, -_3]_{\mathbb{K}} \to [-_1, [-_2, -_3]_{\mathbb{K}}]_{\mathbb{K}}$ *where both sides are considered as functors from* $\mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$

Proof. Let A, B, C be $\mathbb{K}$-vector spaces. We define $\theta : f(a, b) = c \to \theta[f](a)(b) = c$. It is then trivial to check that this is well defined, invertible, linear in f, a, b, and respects precomposition in a, b, and postcompostion in f/c (in the senses used to define the way both functors act on morphisms) , and therefore defines a natural isomorphism of functors from $[-_1 \otimes_{\mathbb{K}} -_2, -_3]_{\mathbb{K}} \to [-_1, [-_2, -_3]_{\mathbb{K}}]_{\mathbb{K}} : \mathcal{D}^{op} \times \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{D}$. From there, one can check that when all three arguments A, B, C of the functors are $\mathbb{K}[G]$-modules, then θ turns out be a $\mathbb{K}[G]$-module isomorphism where the action on both sides can be derived from the definitions of $[,]_{\mathbb{K}}, \otimes_{\mathbb{K}}$. $(a, b, c \to a \cdot g^{-1}, b \cdot g^{-1}, g \cdot c)$. □

Proposition 1.9. *Let* V, W *be* $\mathbb{K}[G]$*-modules, and let* $\phi \in [V, W]^G_{\mathbb{K}}$ *be a* $\mathbb{K}[G]$*-module homomorphism such that* $\operatorname{im}(\phi) \cong W_1 \times W_2$ *for nonzero* $\mathbb{K}[G]$*-submodules* W_1, W_2 *of* $\operatorname{im}(\phi)$*. Then* $\dim([V, W]^G_{\mathbb{K}}) \geq 2$.

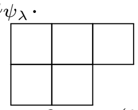
Proof. Let $p_1 : W_1 \times W_2 \to W_1$ and $p_2 : W_1 \times W_2 \to W_2$ be the projection maps. Then we have $\dim([V, W]^G_{\mathbb{K}}) \geq \dim([V, W_1 \times W_2]^G_{\mathbb{K}}) = \dim([V, W_1]^G_{\mathbb{K}} \times [V, W_2]^G_{\mathbb{K}}) = \dim([V, W_1]^G_{\mathbb{K}}) + \dim([V, W_2]^G_{\mathbb{K}}) \geq \dim((p_1 \circ \phi)) + \dim((p_2 \circ \phi)) = 2$. Where the first equality holds because $[V, -]_{\mathbb{K}}, (-)^G$ are both right adjoint functors, $\times$ is a categorical product (which is a limit), and right adjoint functors preserve limits. □

Corollary 1.10. *If* $\mathbb{K}$ *has characteristic 0 and* G *is a finite group, then if* $\dim([V, W]^G_{\mathbb{K}}) = 1$*, then the image of any nonzero* $\mathbb{K}[G]$*-module homomorphism* $\phi : V \to W$ *is irreducible.*

2 Construction of irreducible representations (irreps) of S_n

Recall that irreps of S_n can be indexed by partitions λ of n, which are defined to be weakly decreasing sequences of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ with $\sum_{i=1}^n \lambda_i = n$.

For demonstration purposes, we will take $n = 5$, take $\lambda = (3, 2)$, the young diagram of λ is shown below. We will construct the irreps of S_n from the partitions λ of n by defining two S_n representations $H(\lambda)$ and $E(\lambda)$, and a matrix giving a map $\psi_\lambda : E(\lambda) \to H(\lambda)$. The irreducible representation contructed will then be im_{ψ_λ}.



Define $H(\lambda)$ such that it has a basis $\{v_w\}$ where w's are strings with λ_1 1's, λ_2 2's, etc. In our example,

$$H(3, 2) = \langle v_{11122}, v_{11212}, v_{11221}, v_{12112}, v_{12121}, v_{12211}, v_{21112}, v_{21121}, v_{21211}, v_{22111} \rangle.$$

S_5 acts by sending each basis vector to another basis vector, permuting entries of the string. For example, (13) acts on v_{21112} on the left by swapping the values of the 1st and 3rd places, resulting in v_{11212}. Note that $H(\lambda) = \mathbb{C}[S_n] \otimes_{\mathbb{C}[s_\lambda]} \mathbb{C}$. And the regular map is $H(1, 1, 1, 1, 1)$ with basis $\{v_{12345}, v_{21345}, \dots\}$.

Similarly, define $E(\lambda)$ such that it has a basis $\{u_w\}$ where w's are strings with as many 1's as boxes in the first column, as many 2's as boxes in the second column, and so on. Here S_n acts by sending each basis vector to $\pm$ another basis vector, permuting entries and multiplying by the sgn(perm). For example,

$$E(3, 2) = \langle v_{11223}, v_{11232}, v_{11322}, v_{12123} \rangle.$$

$$(13) \cdot v_{11223} = -v_{21123}, \text{ negative since (13) is odd.}$$

$$(123) \cdot v_{11223} = v_{21123}, \text{ positive since (123) is even.}$$

Remark 2.1.

$$E(\lambda) = \mathbb{C}_{sgn} \otimes_{\mathbb{C}} H(\lambda^t), \text{where } \lambda^t \text{ is the transpose of } \lambda, \text{ and } \otimes_{\mathbb{C}} \text{ is defined as in section 1}$$

Remark 2.2. For all partitions λ, $H(\lambda)^* \cong H(\lambda), \mathbb{C}_{sgn}^* \cong \mathbb{C}_{sgn}$ defined in section 1.

Define a matrix $M_\lambda : E(\lambda) \to H(\lambda)$ (i.e. columns indexed by words corresponding to basis elements of $H(\lambda^t)$ and rows of indexed by words corresponding to basis elements of $H(\lambda)$.

	11223	11232	11322	12123
11122	0	0	0	0
11212	0	0	0	0
11221	0	0	0	1
...				

.

Here, the entry in the column with word w and row with word x is $\begin{cases} 0, & \text{if there are identical columns in } \frac{w}{x}. \\ \pm 1, o.w. & \text{(all columns are distinct).} \end{cases}$

Example 2.3. $\frac{11223}{11122}$ has entry 0 but $\frac{12312}{11122}$ does not.

For ± 1, we determined the sgn from the permutation giving $\frac{w}{x}$ to $\frac{12312}{11122}$.

Remark 2.4. For any two partitions λ^t,μ of n, the action of S_n on the basis elements of $H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)$ is identical to the action of S_n on the set of word pairs $\frac{w}{x}$ by permuting the columns, where w and x are words corresponding to basis elements of $H(\lambda^t)$ and $H(\mu)$ respectively, and the isomorphism is given by $\frac{w}{x} \to v_w \otimes_{\mathbb{C}} v_x$.

Proposition 2.5. *Let w and x be words corresponding to basis vectors of $H(\lambda^t)$ and $H(\lambda)$ respectively. If the word pair $\frac{w}{x}$ has distinct columns (no column appears more than once), then all such word pairs have the same set of columns as $\frac{w}{x}$. Or in a fancier language, when S_n acts on $\frac{w}{x}$ by simultaneously permuting w and x, there is exactly one free orbit of S_n. (**Free** means the stabilizer $= \{identity\}$)*

Proof. We have to show for any two of these word pairs both with all columns distinct, $\frac{w_1}{x_1}, \frac{w_2}{x_2}$, have the same set of columns. Since the set of columns of both word pairs is all that is important for this claim, by permuting the columns, we can reduce to the case where the entries of x_1, x_2 are weakly increasing and therefore both equal to the unique x with weakly increasing entries. If λ is the empty partition, our claim holds vacuously, since both column sets are empty. We now proceed by induction on the length (number of parts) of λ. We have shown the base case holds vacuously (length 0), so it suffices to show any counterexample to this claim could be used to generate another counterexample of strictly smaller length. Observe that the first λ_1 entries of w_1 and w_2 must be distinct, since otherwise, two identical entries of w would lie above 1's from x, resulting in two identical columns of $\frac{w}{x}$. Furthermore, since the w's have λ'_1 1s, and so on, and the length of λ' is exactly λ_1, we see that the first λ_1 entries of both w_1 and w_2 must be precisely the numbers $1, \ldots, \lambda_1$. So for $\frac{w_1}{x}$ and $\frac{w_2}{x}$ to have distinct column sets, the remaining column sets (which all have corresponding x entries at least 2), must be distinct. From this, we define x' with entries $x'_j = x_{\lambda_1 + j} - 1$, and w' with entries $w'_j = w_{j+\lambda_1}$ for $w = w_1, w_2$ and for $1 \leq j \leq (n - \lambda_1)$. We then see that if $\lambda_{>1} := (\lambda_2 \ldots \lambda_l)$ (which has length $l - 1 < l$), then x' is a word corresponding to $\lambda_{>1}$, and w'_1, w'_2 are words corresponding to $\lambda'_{>1}$ with distinct sets of columns, which would be a counterexample of strictly smaller length. This concludes the proof (by contradiction). $\square$

Proposition 2.6. *Let w and x be words corresponding to basis elements of $H(\lambda^t)$ and $H(\mu)$ respectively and let $\frac{w}{x}$ the corresponding word pair. Then unless λ dominates μ, the word pair has a repeated column.*

Proof. Suppose there exist w, x, such that λ doesn't dominate μ but $\frac{w}{x}$ has distinct columns. Without loss of generality, we can assume x is weakly increasing. Then for all integers $1 \leq k \leq \lambda_1^t$, we know that the number of i's appearing in the first $\mu_1 + \ldots \mu_k$ entries of w is at most $\min(\lambda_i^t, k)$ i's for every $1 \leq i \leq \lambda_1 = l(\lambda^t)$. Summing up both sides, we see that $\forall 1 \leq k \leq \lambda_1^t$, $\min(\lambda_1^t, k) + \ldots \min(\lambda_{\lambda_1}^t, k) \leq \mu_1 + \ldots \mu_k$. Now, looking at the young diagram of λ^t, it becomes clear that the left hand side is equal to $\lambda_1 + ... + \lambda_k$ $\forall 1 \leq k \leq \lambda_1^t = l(\lambda)$. But then, the inequality between the left and right hand sides is exactly the statement that λ dominates μ, which contradicts our assumption that the columns of $\frac{w}{x}$ were distinct. □

Corollary 2.7. *From the last two propositions and the remark immediately preceding them, it follows that for any two partitions λ, μ of n, if λ doesn't dominate μ, then every basis element of $H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)$ is fixed by some transposition $t \in S_n$, and that if $\lambda = \mu$, any basis vector not contained in the unique free orbit of the action of S_n on the basis vectors of $H(\lambda^t) \otimes_{\mathbb{C}} H(\lambda)$ is fixed by some transposition $t \in S_n$.*

Lemma 2.8. *Let λ, μ be partitions of n. Then $[E(\lambda), H(\mu)]_{\mathbb{C}} \cong [H(\lambda^t) \otimes_{\mathbb{C}} H(\mu), \mathbb{C}_{sgn}]_{\mathbb{C}}$ as $\mathbb{C}[S_n]$-modules. In particular, this will still hold after taking S_n invariants on both sides and so defines an isomorphism of vector spaces of S_n-equivariant maps.*

Proof. $[E(\lambda), H(\mu)]_{\mathbb{C}} \cong [H(\lambda^t) \otimes_{\mathbb{C}} \mathbb{C}_{sgn}, H(\mu)]_{\mathbb{C}} \cong [H(\lambda^t) \otimes_{\mathbb{C}} \mathbb{C}_{sgn}, H(\mu)^{**}]_{\mathbb{C}} = [H(\lambda^t) \otimes_{\mathbb{C}} \mathbb{C}_{sgn}, [H(\mu)^*, \mathbb{C}]_{\mathbb{C}}]_{\mathbb{C}} \cong [H(\lambda^t) \otimes_{\mathbb{C}} \mathbb{C}_{sgn} \otimes_{\mathbb{C}} H(\mu)^*, \mathbb{C}]_{\mathbb{C}} \cong [H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)^* \otimes_{\mathbb{C}} \mathbb{C}_{sgn}, \mathbb{C}]_{\mathbb{C}} \cong [H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)^*, [\mathbb{C}_{sgn}, \mathbb{C}]_{\mathbb{C}}]_{\mathbb{C}} = [H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)^*, \mathbb{C}_{sgn}^*]_{\mathbb{C}} \cong [H(\lambda^t) \otimes_{\mathbb{C}} H(\mu), \mathbb{C}_{sgn}]_{\mathbb{C}}$. □

Remark 2.9. One can verify that chasing a linear map $f : H(\lambda^t) \otimes_{\mathbb{C}} H(\mu) \to \mathbb{C}_{sgn}$ back through the isomorphisms gives the map from $E(\lambda) \to H(\mu)$ that, when written in matrix form with rows and columns indexed by the bases of $H(\mu)$ and $E(\lambda)$ respectively (or equivalently the words corresponding to the basis vectors of $H(\mu)$ and $H(\lambda^t)$ respectively), has x, w entry equal to $f(v_w \otimes v_x)$. Since we know that this is actually an isomorphism of $\mathbb{C}[S_n]$-modules, if f is an S_n-equivariant map, so is the corresponding matrix.

Proposition 2.10. $\dim([E(\lambda), H(\mu)]) = \dim([H(\lambda^t) \otimes_{\mathbb{C}} H(\mu), \mathbb{C}_{sgn}]_{\mathbb{C}}^{S_n}) = n_f$*, where n_f is the number of free orbits of the S_n on the set of basis vectors of $H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)$. In particular, the dimension is* 0 *unless λ dominates μ, and it's* 1 *if $\lambda = \mu$.*

Proof. One can see that any S_n-equivariant map $f : H(\lambda^t) \otimes_{\mathbb{C}} H(\mu) \to \mathbb{C}_{sgn}$ is uniquely determined by its values on each $v_w \otimes_{\mathbb{C}} v_x$. Since the action of S_n permutes this basis of $H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)$, and f is S_n-equivariant, it is actually completely determined by its values at an element of each orbit. Furthermore, if any element $v_w \otimes v_x$ of an orbit has nontrivial stabilizer, the corresponding word pair must have at least one repeated column, and so must be fixed by a transposition $t \in S_n$. From this it follows that $f(v_w \otimes v_x) = f(t \cdot (v_w \otimes v_x)) = -f(v_w \otimes v_x)$ and therefore $f(v_w \otimes v_x) = 0$. So f is completely determined by its values on a single basis vector from each free orbit. One can also see without much difficulty that any sequence of values $x_1 \ldots x_{n_f} \in \mathbb{C}$ can arise in this way, where n_f is the number of free orbits of the action of S_n on the basis vectors $v_w \otimes v_x$ of $H(\lambda^t) \otimes_{\mathbb{C}} H(\mu)$. From this, we see that $\dim([H(\lambda^t) \otimes_{\mathbb{C}} H(\mu), \mathbb{C}_{sgn}]_{\mathbb{C}}^{S_n}) = n_f$ □

Remark 2.11. Let $v_{x_0} \otimes_{\mathbb{C}} v_{w_0}$ be the unique basis vector of $H(\lambda) \otimes_{\mathbb{C}} H(\lambda^t)$ such that $\frac{w}{x}$ is strictly increasing with respect to the lexicographic order on the columns ($\frac{i_w}{i_x} <_{lex} \frac{j_w}{j_x}$ iff $i_x < j_x$ or $i_x = j_x, i_w < j_w$).

From the above proof one sees that every S_n equivariant map $f : H(\lambda^t) \otimes_{\mathbb{C}} H(\lambda) \to \mathbb{C}_{sgn}$ is of the form $f(v_w \otimes v_x) = \begin{cases} 0, \text{ if there are identical columns in } \frac{w}{x}. \\ \operatorname{sgn}(\pi) z, \text{ if } v_w \otimes_{\mathbb{C}} v_x = \pi \cdot (v_{w_0} \otimes_{\mathbb{C}} v_{x_0}). \end{cases}$ for some $z \in \mathbb{C}$. Then, from the remark directly preceding that proof, we find that every $\psi_\lambda \in [E(\lambda), H(\lambda)]_{\mathbb{C}}^{S_n}$ is of the form zM_λ for some $z \in \mathbb{C}$. So $M_\lambda : E(\lambda) \to H(\lambda)$ is **the** (unique up to scaling) nonzero S_n-equivariant map from $E(\lambda)$ to $H(\lambda)$.

Proposition 2.12. *The image of any nonzero S_n-equivariant map from $E(\lambda) \to H(\lambda)$ is isomorphic to an irreducible representation V_λ depending only on λ, and furthermore, $V_\lambda \cong V_\mu$ if and only if $\lambda = \mu$.*

Proof. The first part of this claim follows directly from corollary 1.10. For the second part of this claim, suppose there is an isomorphism $\phi : V_\lambda \to V_\mu$. Then we have nonzero S_n-equivariant maps $\phi \circ M_\lambda \in [E(\lambda), H(\mu)]_{\mathbb{C}}^{S_n}, \phi^{-1} \circ M_\mu \in [E(\mu), H(\lambda)]_{\mathbb{C}}^{S_n}$, and so both $[E(\lambda), H(\mu)]_{\mathbb{C}}^{S_n}, [E(\mu), H(\lambda)]_{\mathbb{C}}^{S_n}$ have dimension at least 1. But as we have shown previously, this could only be true if λ and μ dominate eachother, which implies that $\lambda = \mu$. $\square$

This gives a construction of irreducible representations V_λ of S_n for each partition λ of n which produces distinct irreducible representations for distinct partitions. Furthermore, from the character theory of finite groups, we see that since the nonisomorphic irreducible representations of S_n are in bijection with conjugacy classes of S_n, which in turn are in bijection with the partitions λ of n, and we were able to construct an injection from the set of partitions to the set of nonisomorphic irreducible representations of S_n, this construction must actually produce all nonisomorphic representations of S_n.

Remark 2.13. Although this is not important for the proof, one can slightly sharpen the proofs of propositions 2.6 and 2.7 to show that in general, the number of free orbits n_f (as defined in proposition 2.10) is equal to the number of young tableaux with shape μ and content λ^t with strictly increasing rows.

3 Basis of irreducible representations (irrep) of S_n

We know that the images of the basis vectors v_w of $E(\lambda)$ by the map M_λ span image. However, their images will not, in general, be linearly independent. However, it is definitely concievable that some subcollections of the basis vectors of $E(\lambda)$ could have images by M_λ that do form a basis. So, one can ask: How many such subcollections of the basis vectors of $E(\lambda)$ are there? This question is more combinatorial in nature than the more representation theoretic questions we started out with. To develop a more general framework for these kinds of questions, we are led to the definition of a matroid. At an intuitive level, these are spaces with a notion of linear dependence of sets of vectors.
There are many equivalent definitions of (finite) matroids. Here is one definition that fits the heuristics above the best.

Definition 3.1. A matroid is a pair of sets $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a finite set and $\mathcal{B}$ is a collection of subsets of $\mathcal{E}$, with following properties:
(B1) $\mathcal{B}$ is not empty.
(B2) (basis exchange property) For $A, B \in \mathcal{B}$ and $a \in A \setminus B$, $\exists b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$

Remark 3.2. Suppose that for $A, B \in \mathcal{B}$, $A \subsetneq B$. Then, $A \setminus B$ is empty. But $B \setminus A$ is not empty. Thus, it follows from the basis exchange property that no member of $\mathcal{B}$ can be properly contained in another.

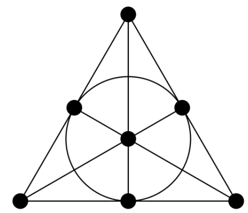


Figure 1: The Fano Plane

Remark 3.3. It is true that if $\mathcal{E}$ is a finite set of an vectorspace V, we can define a matroid structure on $\mathcal{E}$ by setting $\mathcal{B}$ to be the set of maximal linearly independent set in $\mathcal{E}$, these are called **vector matroids**. An important example is the **Fano matroid**. It can be seen as a vector matroid which consists of 7 points in a 3 dimensional vector space over the finite field $\mathbb{Z}_2$.

Remark 3.4. Another type of matroids, that is relevant to our question, is the **graphic matroid**. It is stated as follows. Given a graph (multigraph) G = (E,V), let $\mathcal{E} := E$, and the bases in $\mathcal{B}$ are the spanning forests of G, i.e. spanning edges that do not contain a cycle.

Circling back to the question of how many of these bases there are for V_λ, the answer is not known in general. However, there are nice answers in some situations. As it relates to our problem, the columns of M_λ form a matroid. One thing that can aid our search for a way of counting bases would be to find a "matroid isomorphism" between the matroid formed by the columns of M_λ and some other matroid whose combinatorial properties are better understood. In particular, one of the cases in which there is a nice answer is when $\lambda = (n-1,1)$. In this situation, one can find an isomorphism of matroids between the columns of $M_{(n-1,1)}$, whose maximal independent sets are the bases we are trying to count, and edges of graphs on n labeled vertices, whose maximal independent sets are spanning trees. Therefore, the number of bases in this case is n^{n-2}. In this case one can check that each column of M_λ has n entries total, but only two nonzero entries with opposite signs and that for any such vector v of this form, either v is a column M_λ or $-v$ is a column of M_λ. In the isomorphism, the column whose i-th and j-th entries are nonzero corresponds to the edge between vertex i and vertex j. This map from the set of columns of M_λ to the set of (unordered) pairs of vertices of is invertible and sends linearly dependent sets of column vectors to (edge sets of) subgraphs containing cycles in the labeled graph on n vertices. This is enough to show that it is an isomorphism, since bases of a matroid can be characterized as maximal linearly independent sets of the matroid.