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- Next meeting time: 10am on Friday at Noyes 165

1 Semidirect Product

Definition 1.1 (Inner Semidirect Product)**.** Given a group G with identity element e, a subgroup H, and a normal subgroup $N \trianglelefteq G$, we say that G is the semidirect product of N and H and write $G = N \rtimes H$ if $G = NH$ (product of two subgroups) and $N \cap H = \{e\}$.

Proposition 1.2. *The following statements are equivalent:*

(a) $G = N \rtimes H$.

(b) For every $g \in G$*, there are unique* $n \in N$ *and* $h \in H$ *such that* $g = nh$.

(c) For every $g \in G$*, there are unique* $h \in H$ *and* $n \in N$ *such that* $g = hn$.

(d) The composition $\pi \circ i$ *of the natural embedding* $i : H \to G$ *with the natural projection* $\pi : G \to G/N$ *is an isomorphism between* H *and the quotient group* G/N.

(e) There exists a homomorphism $G \to H$ *that is the identity on* H *and whose kernel is* N*. In other words, there is a split exact sequence of groups:*

$$1 \to N \to G \to H \to 1$$

Proof. (a) implies (b): The existence is obvious. Suppose $g = n_1h_1 = n_2h_2$, then $h_1^{-1}n_1h_1 = h_1^{-1}n_2h_2$, so $h_1^{-1}n_1h_1 = \left(h_1^{-1}n_2h_1\right)\left(h_1^{-1}h_2\right)$. Therefore, $\left(h_1^{-1}n_1h_1\right)\left(h_1^{-1}n_2h_1\right)^{-1} = h_1^{-1}h_2 \in N \cap H = \{e\}$, so $h_1 = h_2$, and the result follows.

(b) implies (c): Let $g \in G$ be given, then $g^{-1} = nh$ for some $n \in N$ and $h \in H$. Thus, $g = h^{-1}n^{-1}$ for some unique $n \in N$ and $h \in H$. Uniqueness follows by way of contradiction.

(c) implies (d): $\pi \circ i$ is clearly a homomorphism. Because $h = he$ trivially, this must be the unique decomposition of h. Since $\pi \circ i(h) = \pi(h) = hN$, if $\pi(h_1) = \pi(h_2)$, then $h_1N = h_2N$, but since $h = he$ is the unique decomposition, it follows that $h_1 = h_2$. For any $gN \in G/N$, since $g = hn$ for some unique n and h, it follows that $gN = hN$, and so $\pi \circ i(h) = gN$. Thus $\pi \circ i$ is an isomorphism.

(d) implies (e): Take the natural projection described in the previous statement.

(e) implies (a): Denote by φ the homomorphism from G to H with the desired property. Let $g \in G$ be given, then $\varphi(g) = h = \varphi(h)$ for some $h \in H$. Thus, $gh^{-1} \in \ker(\varphi) = N$, so $g = nh$ for some $n \in N$ and

$h \in H$, and so $G = NH$. Consider any $a \in N \cap H$, then $\varphi(a) = a$ because $\varphi|_H = \mathrm{id}_H$ but $\varphi(a) = e$ because $a \in \ker(\varphi)$. Thus, $a = e$ and $N \cap H = \{e\}$. $\square$

Definition 1.3 (Outer Semidirect Product)**.** Let us now consider the outer semidirect product. Given any two groups N and H and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$, we can construct a new group $N \rtimes_\varphi H$, called the outer semidirect product of N and H with respect to φ, defined as follows:

- The underlying set is the Cartesian product $N \times H$.
- The group operation $\star$ is determined by the homomorphism $\varphi : (N \rtimes_\varphi H) \times (N \rtimes_\varphi H) \to N \rtimes_\varphi H$:

  $$(n_1, h_1) \star (n_2, h_2) = (n_1 \varphi_{h_1}(n_2), h_1 h_2)$$

  for n_1, n_2 in N and h_1, h_2 in H.

This defines a group in which the identity element is (e_N, e_H) and the inverse of the element (n, h) is $\left(\varphi_{h^{-1}}\left(n^{-1}\right), h^{-1}\right)$. Note how this connects to the inner semidirect product: $\{(e_N, h) \mid h \in H\}$ is a subgroup of $N \rtimes_\varphi H$ and is isomorphic to H; $\{(n, e_H) \mid n \in N\}$ is a normal subgroup of $N \rtimes_\varphi H$ and is isomorphic to N.

Proposition 1.4. *Suppose N and H are subgroups of G such that the decomposition $g = nh$ exists and is unique for all $g \in G$. Let $\varphi : H \to \mathrm{Aut}(N)$ be defined as $\varphi_h(n) = hnh^{-1}$. Then G is isomorphic to the outer semidirect product $N \rtimes_\varphi H$.*

Proof. The isomorphism is given by $\lambda : G \to N \rtimes_\varphi H$ such that $\lambda(g) = \lambda(nh) = (n, h)$. This is well-defined because the decomposition exists and is unique. If $a = n_1 h_1$ and $b = n_2 h_2$, then

$$\begin{aligned} \lambda(ab) &= \lambda(n_1 h_1 n_2 h_2) \\ &= \lambda(n_1 \varphi_{h_1}(n_2) h_1 h_2) \\ &= (n_1 \varphi_{h_1}(n_2), h_1 h_2) \\ &= (n_1, h_1) \star (n_2, h_2) \\ &= \lambda(a) \star \lambda(b), \end{aligned}$$

so λ is a homomorphism and is obviously a bijection. Thus, $G \cong N \rtimes_\varphi H$. $\square$

2 Generality of Tensor Product

Definition 2.1. Let V be a vector space over the field $\mathbb{K}$. The tensor of type (m, n) is the vector space of the form $V^{\otimes m} \otimes (V^*)^{\otimes n}$.

One easily sees from this definition that the following types of tensors can be identified with some common objects/maps in algebra:

- Type $(1, 0)$: vector space.
- Type $(0, 1)$: covectors, also known as the dual space.
- Type $(0, 2)$: bilinear forms on V. This is because $V^* \otimes V^* \cong (V \otimes V)^*$.
- Type $(1, 1)$: linear operators on V.

3 Lie Algebras

Definition 3.1 (Algebra)**.** An algebra $\mathcal{A}$ is a vector space equipped with a bilinear product from $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$.

Definition 3.2 (Lie Algebra)**.** The bilinear product here is this Lie bracket operator $[-,-]$ which satisfies the following properties

1. bilinearity. $[ax+by,z] = a[x,z]+b[y,z]$ and $[x,ay+bz] = a[x,y]+b[x,z]$
2. Alternativity. $[a,a] = 0$
3. Jacobi Identity. $[[a,b],c]+[[b,c],a]+[[c,a],b] = 0$

Using 1 and 2 one can show anticommutivity. $0 = [x+y,x+y] = [x,y]+[y,x]$.

Example 3.3. The trivial Lie bracket: $[-,-] = 0$.

Example 3.4. For an associative algebra $\mathcal{A}$ with bilinear product $(-,-)$, define $[a,b] = (a,b)-(b,a)$. Then $[a,b]$ is bilinear and alternativity follows from the Jacobi Identity.

$$[[a,b],c]+[[b,c],a]+[[c,a],b] = ([a,b],c)-(c,[a,b])+([b,c],a)-(a,[b,c])+([c,a],b)-(b,[c,a]) = 0.$$

$\mathcal{A}$ is called an *enveloping* algebra of $(\mathcal{A},[-,-])$. If we take $\mathcal{A}$ to be the associative algebra of endomorphisms of a $\mathbb{K}$-vector space V with dimension n. i.e $n\times n$ matrices with product defined as matrix multiplication, then the above example is $\mathfrak{gl}_n(\mathbb{K})$ or $\mathfrak{gl}(V)$.

3.1 Derivations

Definition 3.5 (Derivation)**.** A derivation of an algebra $\mathcal{A}$ is an endomorphism $D:\mathcal{A}\to\mathcal{A}$ that satisfies the Leibniz rule

$$D(a\cdot b) = D(a)\cdot b + a\cdot D(b),$$

where $\cdot$ is the bilinear product on $\mathcal{A}$. In the case of Lie Algebra $\mathcal{A}$:

$$D([a,b]) = [D(a),b]+[a,D(b)]$$

Example 3.6. For a Lie Algebra $\mathcal{A}$, fix some $x\in\mathcal{A}$. Then the adjoint mapping $\mathrm{ad}_x(y) = [x,y]$ is a derivation. The Leibniz rule follows from the Jacobi Identity.

$$\begin{aligned}\mathrm{ad}_x([a,b]) &= [x,[a,b]]\\ &= -[[a,b],x]\\ &= [[b,x],a]+[[x,a],b]\\ &= [a,[x,b]]+[[x,a],b]\\ &= [a,\mathrm{ad}_x(b)]+[\mathrm{ad}_x(a),b]\end{aligned}$$

Example 3.7. Let $\mathcal{A}$ be an algebra with bilinear product "$\cdot$" . The set of all derivations $\mathrm{Der}(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$, or $\mathrm{End}(\mathcal{A})$, with the commutator as the Lie bracket. It suffices to show that the commutator of two derivations is a derivation. Suppose $\varphi, \rho \in \mathrm{Der}(\mathcal{A})$, then

$$\begin{aligned}(\varphi\rho - \rho\varphi)(v \cdot w) &= \varphi\rho(v \cdot w) - \rho\varphi(v \cdot w) \\ &= \varphi\left(\rho(v) \cdot w + v \cdot \rho(w)\right) - \rho(\varphi(v) \cdot w + v \cdot \varphi(w)) \\ &= \varphi\rho(v) \cdot w + \cancel{\rho(v) \cdot \varphi(w)} + \cancel{\varphi(v) \cdot \rho(w)} + v \cdot \varphi\rho(w) \\ &\quad - \rho\varphi(v) \cdot w - \cancel{\varphi(v) \cdot \rho(w)} - \cancel{\rho(v) \cdot \varphi(w)} - v \cdot \rho\varphi(w) \\ &= (\varphi\rho - \rho\varphi)(v) \cdot w - v \cdot (\varphi\rho - \rho\varphi)(w)\end{aligned}$$

Hence, $\varphi\rho - \rho\varphi \in \mathrm{Der}(\mathcal{A})$, so $\mathrm{Der}(\mathcal{A})$ is a Lie subalgebra of $\mathrm{End}(\mathcal{A})$.

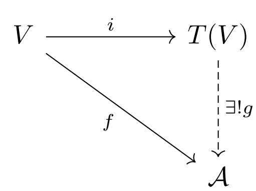
4 The Tensor Algebra

Definition 4.1. For a vector space V, its kth tensor power is defined to be $T^k V = V^{\otimes k}$, and its tensor algebra is defined as $T(V) = \bigoplus_{n \geq 0} T^k V$. The multiplication in $T(V)$ is defined as the "concatenation" of tensor products and then extended linearly to other elements. For example,

$$(e_2 \otimes e_5)(e_1 \otimes e_3 \otimes e_4) = e_2 \otimes e_5 \otimes e_1 \otimes e_3 \otimes e_4.$$

Therefore, $T(V)$ is has a graded algebra structure.

Proposition 4.2. *The tensor algebra satisfies the universal property. Let V be a vector space over field $\mathbb{K}$, let i be the natural inclusion map from V to $T(V)$, and let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra. If there exists a linear map $f : V \to \mathcal{A}$, then there exists a unique algebra homomorphism $g : T(V) \to \mathcal{A}$ such that the following diagram commutes, i.e. $f = g \circ i$.*



Proof. Define g by the following:

$$g(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = f(v_1) \cdot f(v_2) \cdot \ldots \cdot f(v_k),$$

extended linearly to all elements in $T(V)$. The dot on the right hand side is the bilinear product in $\mathcal{A}$, and we may drop the parenthesis because $\mathcal{A}$ is associative. Note that g is linear by definition. g is also obviously an algebra homomorphism. Also note that $f = g \circ i$ because $g \circ i(v) = g(v) = f(v)$ for all $v \in V$. Finally, g is unique because g and f must agree on vectors in V, and because g is an algebra homomorphism,

$$g(v_1 \otimes v_2) = g(v_1 \cdot v_2) = g(v_1)g(v_2) = f(v_1)f(v_2).$$

The same argument can be applied inductively and the uniqueness of g follows. $\square$

In categorical terms, the tensor algebra is a functor from the category of vector spaces to the category of $\mathbb{K}$-algebras.

Definition 4.3 (Two Sided Ideal)**.** Given a algebra $\mathcal{A}$, a subalgebra I is an ideal of $\mathcal{A}$ if for all $a \in \mathcal{A}$ and $x \in I$ the products $a \cdot x, x \cdot a \in I$

Definition 4.4 (Symmetric Algebra)**.** Let V be a vector space and $T(V)$ the associated tensor algebra. Then we can define an ideal I generated by elements of the form $x \otimes y - y \otimes x$. The symmetric algebra of V is defined as $\mathrm{Sym}(V) = T(V)/I$. The kth symmetric power of V is the subspace of $S(V)$ spanned by k-fold symmetric product of vectors in V.

Because we quotient out vectors of the form $x \otimes y - y \otimes x$, it follows that $S(V)$ is isomorphic $\mathbb{K}[B]$, where B is considered as indeterminate. This provides a coordinate-free way polynomial ring over V.