06/01/22 Notes

Dylan Roscow and Zhuo Zhang

June 1, 2022

- Next meeting time: 10am on Thursday at Noyes 165

1 More on Tensor Product

First recall the following proposition:

Proposition 1.1. *Let V, W be finite dimensional vector spaces of dimensions m and n over a field $\mathbb{K}$, then $\mathrm{Hom}(V, W) \cong V^* \otimes W$.*

Proof. Although we can see directly from their dimensions that $\mathrm{Hom}(V, W)$ is isomorphic to $V^* \otimes W$, here we shall construct a more "canonical" map between these two spaces. Take $T \in \mathrm{Hom}(V, W)$ and an arbitrary vector $v \in V$, we have

$$T(v) = a_1^{(v)} w_1 + a_2^{(v)} w_2 + \cdots + a_n^{(v)} w_n,$$

where $a_i^{(v)}$ are scalars depending on v. Note that

$$\sum_{i=1}^{n} a_i^{(u+\lambda v)} w_i = T(u + \lambda v) = T(u) + \lambda T(v) = \sum_{i=1}^{n} \left(a_i^{(u)} + \lambda a_i^{(v)} \right) w_i.$$

It follows that each $a_i \in \mathrm{Hom}(V, \mathbb{K}) = V^*$. Now define a map $\psi : \mathrm{Hom}(V, W) \to V^* \otimes W$ to be:

$$\psi(T) = a_1 \otimes w_1 + a_2 \otimes w_2 + \cdots + a_m \otimes w_m.$$

The claim is that this map is an isomorphism. First, it is linear because if

$$\psi(T) = \sum_{i=1}^{n} a_i \otimes w_i \quad \text{and} \quad \psi(U) = \sum_{i=1}^{n} b_i \otimes w_i,$$

then we have

$$(T + \lambda U)(v) = \sum_{i=1}^{n} \left(a_i^{(v)} + \lambda b_i^{(v)} \right) w_i,$$

$$\Rightarrow \ \psi(T + \lambda U) = \sum_{i=1}^{n} (a_i + \lambda b_i) \otimes w_i = \sum_{i=1}^{n} a_i \otimes w_i + \lambda \left(\sum_{i=1}^{n} b_i \otimes w_i \right) = \psi(T) + \lambda \psi(U).$$

So the map is linear. Next, we show that the map is surjective. Suppose V and W have bases $\{v_1, \cdots, v_m\}$ and $\{w_1, \cdots, w_n\}$, then each basis vector of $V^* \otimes W$ is of the form $v_i^* \otimes w_j$, where $v_i^*(v_k) = \delta_{ik}$. It suffices to show that each basis of this form is hit by ψ. For fixed i and j, consider $T : V \to W$ defined by its action on the basis of V:

$$T(v_k) = \delta_{ik} w_j = v_i^*(v_k) w_j.$$

T is clearly linear. Also notice that $\psi(T) = v_i^* \otimes w_j$, so the map ψ is surjective, which is enough to say that ψ is an isomorphism because $\dim(\mathrm{Hom}(V, W)) = \dim(V^* \otimes W)$ $\square$

One direct consequence of Proposition 1.1 is that, suppose V and W are representations of G, then V^* is also a representation of G. Thus, $\mathrm{Hom}(V, W) \cong V^* \otimes W$ is also a representation of G.

Corollary 1.2. *Let $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ be representations of G, then* $\mathrm{Hom}(V, W)$ *is also a G-module, with the action defined as follows. For $g \in G$ and $T \in \mathrm{Hom}(V, W)$:*

$$(g \cdot T)(v) = \varphi_g T \rho_{g^{-1}}(v).$$

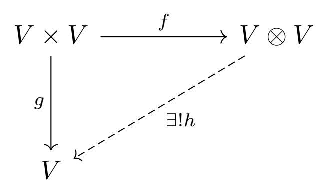
Proof. The action on $\mathrm{Hom}(V, W)$ is clearly linear. It is invertible because action by g^{-1} is its inverse. Finally, the map from G to $\mathrm{Hom}(V, W)$ is a homomorphism because

$$(gh) \cdot T = \varphi_{gh} T \rho_{(gh)^{-1}} = \varphi_g \varphi_h T \rho_{h^{-1}} \rho_{g^{-1}} = g(h \cdot T).$$

$\square$

2 Application: Matrix Multiplication

Note that matrix multiplication is a bilinear map $V \times V \to V$, where V is the vector space of $n \times n$ matrices. Hence, by the universal property of tensor product:



There exists a unique map h:

$$h \in \mathrm{Hom}(V \otimes V, V) \cong (V \otimes V)^* \otimes V \cong V^* \otimes V^* \otimes V \cong V^{\otimes 3}.$$

Hence, the map h, which represents matrix multiplication, is a vector in the vector space $V^{\otimes 3}$. Here we have used the following fact.

Proposition 2.1. *Let V, W be finite dimensional vector spaces of dimensions m and n over a field $\mathbb{K}$, then $(V \otimes W)^* \cong V^* \otimes W^*$.*

Proof. We shall again construct a canonical map. Suppose V and W have bases $\beta = \{v_1, \cdots, v_m\}$ and $\gamma = \{w_1, \cdots, w_n\}$. Define the map $\psi : V^* \otimes W^* \to (V \otimes W)^*$ by its action on the basis of $V^* \otimes W^*$:

$$\psi(f \otimes g)(v \otimes w) = f(v)g(w).$$

ψ is linear by definition. We now show that it is one-to-one. Suppose $\psi(f \otimes g)(v \otimes w) = f(v)g(w) = 0$ for all $v \in V$, $W \in W$. That is, $f \otimes g$ is mapped to the zero function on $V \otimes W$, then we argue that either $f = 0$ or $g = 0$. Suppose that there exists some $v \in V$ such that $f(v) \neq 0$. Since $f(v)g(w_j) = 0$ for all w_j, we conclude that $g(w_j) = 0$, so g is the zero function on W. Thus, $f \otimes g = 0$. The other case can be proved similarly. Hence, $\ker(\psi) = \{0\}$, and the map is one-to-one and thus an isomorphism. $\square$

Remark. The result can be extended to the case where V and W are infinite dimensional. For this proof, see `https://planetmath.org/tensorproductofdualspacesisadualspaceoftensorproduct`.

3 Representation Ring

Definition 3.1. The representation ring of a group G, also called the Grothendieck ring, is the set of isomorphism classes of representations of G with a ring structure. Addition is defined as direct sum, and multiplication is defined as tensor product. More precisely, suppose φ and ρ are two representations of G and $[\cdot]$ is their isomorphism classes, then

$$[\varphi] + [\rho] := [\varphi \oplus \rho] \qquad [\varphi] \cdot [\rho] := [\varphi \otimes \rho].$$

The additive identity of $\mathrm{Rep}(G)$ is the formal zero, so inverse is also defined formally. The multiplicative identity of $\mathrm{Rep}(G)$ is the trivial representation.

Proposition 3.2. *Addition and multiplication in* $\mathrm{Rep}(G)$ *are well-defined.*

Proof. Suppose $\varphi \sim \varphi'$ and $\rho \sim \rho'$, then $\chi_\varphi = \chi_{\varphi'}$ and $\chi_\rho = \chi_{\rho'}$. Therefore,

$$\chi_{\varphi\oplus\rho} = \chi_\varphi + \chi_\rho = \chi_{\varphi'} + \chi_{\rho'} = \chi_{\varphi'\oplus\rho'} \quad \Longrightarrow \quad \varphi \oplus \rho \sim \varphi' \oplus \rho'$$

The same can be proved for multiplication. □

4 Group Product and Their Representation

Given two groups G and H, their Cartesian product $G \times H$ is a group under the operation $(g, h) \cdot (g', h') = (gg', hh')$. Additionally, $1_{G\times H} = (1_G, 1_H)$. Given two representations of G and H, we can construct a representation of $G \times H$ as follows:

Proposition 4.1. *Given representation* $p : G \to \mathrm{GL}(V)$ *and* $q : H \to \mathrm{GL}(W)$*, then* $p \otimes q : G \times H \to \mathrm{GL}(V \otimes W)$*, defined by the action* $(g, h) \cdot (v \otimes w) = (g \cdot v) \otimes (h \cdot w)$ *extended linearly, is a representation.*

Proof. For any $(g, h), (g', h') \in G \times H$:

$$\begin{aligned} ((g, h) \cdot (g', h')) \cdot (v \otimes w) &= ((gg') \cdot v) \otimes ((hh') \cdot w) \\ &= (g \cdot (g' \cdot v)) \otimes (h \cdot (h' \cdot w)) \\ &= (g, h) \cdot ((g', h') \cdot (v \otimes w)) \end{aligned}$$

Therefore, $p \otimes q$ is a representation. □

Lemma 4.2. *For any representations* p *and* q*, we have* $\chi_{p\otimes q}(g, h) = \chi_p(g) \cdot \chi_q(h)$*.*

Proof. As with the tensor product of G-modules, the matrix corresponding to $p\otimes q$ is given by the Kronecker product. Thus, $\chi_{p\otimes q}(g, h) = \mathrm{Tr}((p \otimes q)(g, h)) = \mathrm{Tr}(p(g) \otimes q(h)) = \mathrm{Tr}(p(g)) \cdot \mathrm{Tr}(q(h)) = \chi_p(g) \cdot \chi_q(h)$. □

Proposition 4.3. *If* p *and* q *are irreducible, then* $p \otimes q$ *is irreducible.*

Proof. Since p and q are irreducible, we have $(\chi_p \mid \chi_p) = 1$ and $(\chi_q \mid \chi_q) = 1$. Thus, using the lemma:

$$\begin{aligned}
(\chi_{p\otimes q} \mid \chi_{p\otimes q}) &= \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} |\chi_{p\otimes q}(g,h)|^2 \\
&= \frac{1}{|G|\cdot|H|} \sum_{g\in G}\sum_{h\in H} |\chi_p(g)|^2 \cdot |\chi_q(h)|^2 \\
&= \left(\frac{1}{|G|}\sum_{g\in G} |\chi_p(g)|^2\right)\left(\frac{1}{|H|}\sum_{h\in H} |\chi_q(h)|^2\right) \\
&= (\chi_p \mid \chi_p)(\chi_q \mid \chi_q) = 1\cdot 1 = 1
\end{aligned}$$

Therefore, since $(\chi_{p\otimes q} \mid \chi_{p\otimes q}) = 1$, we have that $p \otimes q$ is irreducible. □

Proposition 4.4. *The converse is also true. That is, each irreducible representation of $G_1 \times G_2$ is of the form $p \otimes q$, where p and q are some irreducible representations of G_1 and G_2, respectively.*

Proof. Let $\chi_1^{(1)}, \cdots, \chi_k^{(1)}$ be a complete set of irreducible characters of G_1, and same for $\chi_1^{(2)}, \cdots, \chi_\ell^{(2)}$. From proposition 4.3 we know that characters of the form $\chi_i^{(1)}(s_1)\chi_j^{(2)}(s_2)$ are irreducible. We want to show that there are no other irreducible characters of $G_1 \times G_2$. Because distinct irreducible characters are orthonormal, it suffices to show that any class function on $G_1 \times G_2$ that is orthogonal to all of $\chi_i^{(1)}(s_1)\chi_j^{(2)}(s_2)$ is zero. To see, suppose f is a class function on $G_1 \times G_2$ such that

$$\sum_{(s_1,s_2)\in G_1\times G_2} f(s_1,s_2)\overline{\chi_i^{(1)}(s_1)\chi_j^{(2)}(s_2)} = 0 \qquad \text{for all } i, j$$

Now, fix $\chi_j^{(2)}$ and s_2 and consider the function h:

$$h(s_1) = \sum_{s_2\in G_2} f(s_1,s_2)\chi_j^{(2)}(s_2)$$

Then h is a class function because

$$h(t_1 s_1 t_1^{-1}) = \sum_{s_2\in G_2} f(t_1 s_1 t_1^{-1}, s_2)\chi_j^{(2)}(s_2) = \sum_{s_2\in G_2} f(s_1,s_2)\chi_j^{(2)}(s_2) = h(s_1)$$

Substituting h into the first equation gives

$$\sum_{s_1\in G_1} h(s_1)\overline{\chi_i^{(1)}(s_1)} = 0 \quad \Rightarrow \quad \left\langle h, \chi_i^{(1)}\right\rangle = 0$$

Since h is a class function on G_1 and is orthogonal to $\chi_i^{(1)}$ for all i, it follows that $h = 0$. Hence, f is orthogonal to all $\chi_j^{(2)}$, but f is a class function on G_2 when s_1 is fixed, and so it follows that $f = 0$, which concludes the proof. □

5 Frobenius Characteristic Map

Definition 5.1. The characteristic vector space R^n of S_n is the set of all formal linear combinations of irreducible characters of S_n. That is:

$$R^n = \left\{ \sum_{\lambda\vdash n} a_\lambda \chi_\lambda \,\middle|\, a_\lambda \in \mathbb{C} \right\}$$

We can gather these together into one large vector space R as follows:

$$R = \bigoplus_{n=0}^{\infty} R^n$$

Next, we wish to put a ring structure on R. We will do this using an induced representation (which will be formally defined later). Consider $\lambda \vdash n$ and $\mu \vdash m$ with their corresponding representations V_λ and V_μ. Then, $V_\lambda \otimes V_\mu$ is an $S_n \times S_m$ representation. We define the product of two characters as follows:

$$\chi_\lambda \cdot \chi_\mu = \chi_W, \qquad W = \mathrm{ind}_{S_n \times S_m}^{S_{n+m}} (V_\lambda \otimes V_\mu)$$

Here, $W \in R^{n+m}$ because it can be decomposed as $\sum_{\nu \vdash n+m} a_\nu V_\nu$. This means that R has the structure of a graded ring (similar to polynomials).

Definition 5.2. Λ is the ring of symmetric formal power series. That is the set of all power series $f(x_1, x_2, \dots)$ such that for any $\sigma \in S_n$, we have $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, x_{n+1}, \dots) = f(x_1, \dots, x_n, x_{n+1}, \dots)$.

Λ^n denotes the set of symmetric formal power series with degree n. We have a similar decomposition:

$$\Lambda = \bigoplus_{n=0}^{\infty} \Lambda^n$$

Λ^n is finite dimensional, with the dimension being the number of partitions of n. The following are two ways to explicitly give a basis of Λ^n. The first is the basis given by the monomial symmetric functions m_λ. For a given partition $\lambda = (\lambda_1, \dots, \lambda_k)$, define m_λ to be the formal sum of all monomials of the form $x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(k)}^{\lambda_k}$ where σ is an arbitrary permutation.

For the other basis of elements h_λ, let h_t be the formal sum of all monomials of degree t. Then, define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$.

Definition 5.3. The Hall inner product on Λ is the inner product such that $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$.

Now, we define the Frobenius Characteristic Map:

Definition 5.4. The Frobenius Characteristic Map $\mathrm{ch} : R \to \Lambda$ is given by $\mathrm{ch}(\chi_\lambda) = s_\lambda$

It can be shown that ch is an isomorphism $\mathrm{ch}(\chi_\lambda \cdot \chi_\mu) = s_\lambda \cdot s_\mu$ and that it is an isometry under the Hall inner product $\langle f, g \rangle = \langle \mathrm{ch}(f), \mathrm{ch}(g) \rangle$.