06/22/22 Notes

Zhuo Zhang and Dylan Roscow

June 22, 2022

1 Littlewood-Richardson Coefficients

The Littlewood-Richardson coefficients appear in many different contexts and have very rich combinatorial meanings.

Let groups G_1 and G_2 be given. Recall that the group product $G_1 \times G_2$ is also a group, whose irreducible representations are exactly $\rho^{(1)} \otimes \rho^{(2)}$, where $\rho^{(i)}$ is are irreducible representations of G_i.

Let λ and μ be partitions of natural numbers m and n, respectively. Let V^λ and V^μ be the corresponding Specht modules (irreducible representations of the symmetric group S_n). Clearly there exists a natural inclusion map $i : S_m \times S_n \to S_{m+n}$, and so we have the induced representation $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \left(V^\lambda \otimes V^\mu\right)$. To obtain the decomposition of this representation, we have

$$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \left(V^\lambda \otimes V^\mu\right) = \bigoplus_{\nu \,\vdash\, m+n} c_{\lambda\mu}^{\nu} V^\nu.$$

Here, the non-negative integers $c_{\lambda\mu}^{\nu}$ are called the Littlewood-Richardson coefficients. We may also obtain a formula to describe these coefficients more explicitly:

$$c_{\lambda\mu}^{\nu} = \dim \mathrm{Hom}\left(V^\nu, \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \left(V^\lambda \otimes V^\mu\right)\right).$$

By the Frobenius reciprocity formula, we also have the following equivalent statement:

$$c_{\lambda\mu}^{\nu} = \dim \mathrm{Hom}\left(V^\lambda \otimes V^\mu, \mathrm{Res}_{S_m \times S_n}^{S_{m+n}} V^\nu\right).$$

The same Littlewood-Richardson coefficients also appear in the representation of GL_n, the general linear groups. Recall that the irreducible polynomial representations of $GL_n(\mathbb{C})$ are indexed by paritions of length at most n. Let V^λ and V^μ be two irreducible polynomial representations of $GL_n(\mathbb{C})$, then

$$V^\lambda \otimes V^\mu = \bigoplus_{\ell(\nu) \leq n} c_{\lambda\mu}^{\nu} V^\nu,$$

or equivalently,

$$c_{\lambda\mu}^{\nu} = \dim \mathrm{Hom}\left(V^\nu, V^\lambda \otimes V^\mu\right).$$

Moreover, the $c_{\lambda\mu}^{\nu}$'s also appears as the coefficients in expressing the product of two Schur polynomial as linear combinations of other Schur polynomials.

The Littlewood-Richardson coefficients are also related to the following Horn's conjecture. Given three Hermitian matrices A, B, C such that $A + B = C$ with the eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$, and $\nu = (\nu_1, \ldots, \nu_n)$ respectively. Then, for three partitions (λ, μ, ν) there are such matrices with these eigenvalues if and only if $c^{\nu}_{\lambda\mu} > 0$. The final lemma for the proof of Horn's Conjecture is the following theorem:

Theorem 1.1 (Knutson-Tao Saturation Theorem)**.** *For partitions λ, μ, ν, $c^{\nu}_{\lambda\mu} > 0$ if and only if $c^{N\nu}_{N\lambda,N\mu} > 0$ for all $N \geq 1$, where $N\alpha = (N\alpha_1, \ldots, N\alpha_n)$.*

Another related theorem in Kostant's Conjecture which was later proven by Berenstein and Zelevinsky.

Theorem 1.2. *For a fixed n, let $\rho = (n, n-1, n-2, \ldots, 1)$. If μ is a partition such that $\ell(\mu) \leq n$ and $\mu \leq 2\rho$ in dominance order, then $c^{\mu}_{\rho\rho} > 0$.*

We are looking at a generalization of this conjecture:

Conjecture 1.3 (Generalized Kostant's Conjecture)**.** *For a fixed n, again let $\rho = (n, n-1, n-2, \ldots, 1)$. For any $N \geq 1$, if μ is a partition such that $\ell(\mu) \leq n$ and $\mu \leq 2N\rho$ in dominance order, then $c^{\mu}_{N\rho,N\rho} > 0$.*

The following theorem provides a way to compute the Littlewood-Richardson coefficients.

Theorem 1.4 (Littlewood-Richardson Rule)**.** *Let λ, μ, ν be given partitions. The Littlewood-Richardson Rule states that the coefficient $c^{\nu}_{\lambda\mu}$ is equal to the number of tableaux with the following properties: first, it has shape ν/λ and content μ; second, it is semistandard, which is to say that the rows are weakly increasing and columns are strictly increasing; finally, the ballot condition has to be satisfied: for any initial part of the sequence obtained by concatenating the reversed rows, the number of i's must be greater than or equal to the number of $i+1$'s.*