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1 Induced Representation

Let $\varphi : G \to GL(V)$ be a representation of G. Let H be a subgroup of G and let W be a subspace of V that is H-invariant. That is, $\varphi_h(W) = W$ for all $h \in H$. Denote by $\theta : H \to GL(W)$ the representation of H in W thus defined.

Consider $s \in G$, the subspace $\varphi_s(W)$ is completely determined by which coset of H contains s. This is so because if $s \equiv t \pmod{H}$, then $s = th$ for some $h \in H$. Thus $\varphi_s(W) = \varphi_{th}(W) = \varphi_t\varphi_h(W) = \varphi_t(W)$ because W is H-invariant. Let σ be a coset of H in G, then define W_σ to be $\varphi_s(W)$ for any $s \in \sigma$ (which is well defined as outlined above). It is clear that the W_σ are permuted among themselves by the φ_g, for any $g \in G$. Hence, the sum $\sum\limits_{\sigma \in G/H} W_\sigma$ is a subrepresentation of G.

Definition 1.1. We say that the representation ρ of G in V is induced by the representation θ of H in W if V is equal to the sum of the W_σ ($\sigma \in G/H$) and if that sum is direct. That is: $V = \bigoplus\limits_{\sigma \in G/H} W_\sigma$.

This can be reformulated in several ways:

1. Each $x \in V$ can be written uniquely as $x = \sum\limits_{\sigma \in G/H} x_\sigma$, with $x_\sigma \in W_\sigma$.

2. If R is a system of representatives of G/H, then V is the direct sum of $\varphi_r(W)$, with r ranging in R.

In particular, we have that

$$\dim(V) = \sum_{r \in R} \dim(\varphi_r(W)) = [G : H] \dim(W)$$

Remark 1.2. Here we list and explain several important facts/examples about induced representation.

1. Let $\rho : G \to GL(V)$ be the regular representation of G. Recall that V has basis $\{e_t\}_{t \in G}$ and $\rho_s(e_t) = e_{st}$. Let H be a subgroup of G and W is spanned by $\{e_h\}_{h \in H}$. We argue that the representation $\theta : H \to GL(W)$ induces ρ. First, let σ be a coset of H, then by definition $W_\sigma = \rho_s(W)$ for any $s \in \sigma$. Therefore, W_σ is spanned by $\{e_{sh}\}_{h \in H} = \{e_t\}_{t \in \sigma}$. Therefore, the bases of W_σ are disjoint and their union is the basis of V. Hence, $V = \bigoplus\limits_{\sigma \in G/H} W_\sigma$ and ρ is induced by θ.

2. Let V be spanned by $\{e_\sigma\}_{\sigma \in G/H}$. Define ρ (or the action of G on V) by $\rho_g(e_\sigma) = e_{g\sigma}$. This is well defined because if $\sigma \in G/H$, so is $g\sigma$. Note that the vector e_H is invariant under H because $\rho_h(e_H) = e_{hH} = e_H$ for all $h \in H$. Define W to be the span of e_H, so $\theta : H \to GL(W)$ is a

representation of H. Under this definition, $W_\sigma = \text{span}(\rho_s(e_H)) = \text{span}(e_{sH}) = \text{span}(e_\sigma)$ for any $s \in \sigma$. Thus, each W_σ has basis $\{e_\sigma\}$ and their disjoint union is the basis of V. Hence, $V = \bigoplus_{\sigma \in G/H} W_\sigma$ and ρ is induced by θ.

3. If $\rho^{(1)}$ is induced by $\theta^{(1)}$ and $\rho^{(2)}$ is induced by $\theta^{(2)}$, then $\rho^{(1)} \oplus \rho^{(2)}$ is induced by $\theta^{(1)} \oplus \theta^{(2)}$.

4. Let $\rho : G \to GL(V)$ be induced by $\theta : H \to GL(W)$. Now we find a subspace $W' \subset W$ such that W' is stable under H. By the discussions in section 2, we know that

$$V' = \sum_{\sigma \in G/H} W'_\sigma = \sum_{r \in R} \rho_r(W)$$

is stable under G. The question is whether V' is induced by W'. We already know that the sum $V = \bigoplus_{\sigma \in G/H} W_\sigma$ is direct. Since $W' \subset W$, we must have that $W'_\sigma = \rho_s(W') \subset \rho_s(W) = W_\sigma$ (for any $s \in \sigma$), so the sum $V' = \bigoplus_{\sigma \in G/H} W'_\sigma$ is also direct, and so the representation of G in V' is induced by the representation of H in W'.

5. If ρ is induced by θ, if ρ' is a representation of G, and if ρ'_H is the restriction of ρ' to H, then $\rho \otimes \rho'$ is induced by $\theta \otimes \rho'_H$.

This can be reformulated as:

$$\operatorname{Ind}_H^G \left(W \otimes_{\mathbb{C}} \operatorname{Res}_H^G V' \right) \ \cong_G \operatorname{Ind}_H^G W \otimes_{\mathbb{C}} V'.$$

2 Existence and Uniqueness of Induced Representation

Lemma 2.1. *Suppose that (V, ρ) is induced by (W, θ). Let $\rho' : G \to GL(V')$ be a linear representation of G, and let $f : W \to V'$ be a linear map such that $f(\theta_t w) = \rho'_t f(w)$ for all $t \in H$ and $w \in W$. Then there exists a unique linear map $F : V \to V'$ which extends f and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$.*

Proof. First assume that F exists. Suppose $x \in W_\sigma$ and pick any $s \in \sigma$, then $x \in \rho_s(W)$ and so $\rho_s^{-1}(x) \in W$. Thus,

$$F(x) = \rho'_s \circ F \circ \rho_s^{-1}(x) = \rho'_s \circ f \circ \rho_s^{-1}(x),$$

because F extends f on W. Therefore, the value of F on $x \in \rho_s(W)$ is determined. But since V is a direct sum of $\rho_s(W)$, where s ranges over a transversal of G/H, it follows that the value of F on V is determined. This proves the uniqueness of F. To prove that such a map exists, we shall define it to be this way. Suppose $x \in W_\sigma$ and pick any $s \in \sigma$, define $F(x) = \rho'_s \circ f \circ \rho_s^{-1}(x)$. We need to verify two things: F is well-defined (doesn't depend on which $s \in \sigma$ we pick); F extends f and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$.

First, F doesn't depend on which $s \in \sigma$ we pick because, if we replace s by sh for some $h \in H$, then

$$\begin{aligned} \rho'_{sh} \circ f \circ \rho_{(sh)^{-1}}(x) &= \rho'_s \rho'_h \circ f \circ \theta_{h^{-1}} \rho_s^{-1}(x) \\ &= \rho'_s \rho'_h \rho'_{h^{-1}} \circ f \circ \rho_s^{-1}(x) \\ &= \rho'_s \circ f \circ \rho_s^{-1}(x). \end{aligned}$$

F extends f obviously, because if $x \in W$, then the coset $\sigma = W$, so $s \in H$, and by the same reasoning above we know that $F(x) = f(x)$. Finally, $F \circ \rho_s = \rho'_s \circ F$ holds for all $s \in G$. This is NOT by definition because x may not be in W_{sH}. However, suppose $x \in W_\sigma$, where $\sigma = tH$, then

$$\rho'_s \circ F(x) = \rho'_s \rho'_t \circ f \circ \rho_t^{-1}(x).$$

On the other hand, since $x \in W_{tH}$, we have $\rho_s(x) = W_{stH}$. Therefore,

$$F \circ \rho_s(x) = \rho'_{st} \circ f \circ \rho_{(st)^{-1}}(\rho_s(x)) = \rho'_s \rho'_t \circ f \circ \rho_t^{-1}(x).$$

Therefore, $F \circ \rho_s = \rho'_s \circ F$ holds for all $s \in G$. □

Theorem 2.2. *Let (W, θ) be a linear representation of H. There exists a linear representation (V, ρ) of G which is induced by (W, θ), and it is unique up to isomorphism.*

Proof. Let us first prove the existence of the induced representation ρ. In view of example 3, above, we may assume that θ is irreducible (if not, then decompose θ into irreducibles and induct on each of them). In this case, θ is isomorphic to a subrepresentation of R_H, the regular representation of H, because

$$R_H \cong d_1\varphi^{(1)} \oplus \cdots \oplus d_k\varphi^{(k)}$$

where $\varphi^{(i)}$, d_i are the irreducible representations and their dimensions. By example 1, R_H which can be induced to the regular representation of G. Applying example 4, we conclude that θ itself can be induced.

Next, we show the uniqueness. Suppose (V, ρ) and (V', ρ') are both induced by (W, θ). Let $i : W \to V'$ be the natural inclusion map, then i is clearly an intertwiner. By Lemma 1, there exists a unique intertwiner $F : V \to V'$ that extends i, and it must be defined by

$$F(x) = \rho'_s \circ i \circ \rho_s^{-1}(x) = \rho'_s \rho_s^{-1}(x) \qquad \text{if } x \in W_{sH}.$$

Since $\rho_s^{-1}(x) \in W$, the image of F contains all $\rho'_s(W)$, where s is in the transversal of G/H. Therefore, $\operatorname{im} F = V'$, and F is thus an isomorphism since V and V' have the same dimension. □

3 Induced Characters

Since induced representation of $H \leq G$ is unique up to isomorphism, we should be able to determine its character uniquely from H. This is done in the next theorem.

Theorem 3.1. *Let R be a system of representatives of G/H. For each $u \in G$, we have*

$$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}us \in H}} \chi_\theta(s^{-1}us)$$

Proof. V is the direct sum of $\rho_r W$, with r ranging in R. Consider the transformation ρ_u on each of these subspaces. Suppose $ur = r_u t$ with $t \in H$, then $\rho_u(\rho_r W) = \rho_{r_u} W$. To obtain a basis of V, take a basis from each $\rho_r W$ and take their (disjoint) union. If $r_u \neq r$, then $\rho_u(\rho_r W) \cap \rho_r W = \{0\}$, so the corresponding diagonal entries must be zero. If $r_u = r$, the trace of the corresponding small matrix is what we want. Thus,

$$\chi_\rho(u) = \sum_{r \in R_u} \operatorname{tr}_{\rho_r W}(\rho_{u,r}),$$

where R_u is the set of r such that $r_u = r$, and $\rho_{u,r}$ is the restriction of ρ_u to $\rho_r W$. Note that $r \in R_u$ if and only if $ur = rt$ for some $t \in H$, that is, $r^{-1}ur \in H$. Note that ρ_r defines an isomorphism between W and $\rho_r W$. Moreover, denote $t = r^{-1}ur \in H$, then

$$\rho_r \circ \theta_t = \rho_r \circ \rho_{r^{-1}ur} = \rho_{u,r} \circ \rho_r$$

Therefore, we have

$$\operatorname{tr}_{\rho_r W}(\rho_{u,r}) = \operatorname{tr}_{\rho_r W}(\theta_t) = \chi_\theta(t)$$

Hence,

$$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta\left(r^{-1}ur\right)$$

The second formula follows because, if $s = rt$ for $t \in H$, then

$$\chi_\theta\left(s^{-1}us\right) = \chi_\theta\left(t^{-1}r^{-1}urt\right) = \chi_\theta\left(r^{-1}ur\right)$$

because χ_θ is a class function on H. $\square$

Remark 3.2. We provide another proof of the formula, which is similar to Serre's approach but is more straightforward and heuristic. For the sake of convenience, let $\{w_1, \cdots, w_n\}$ be a basis for W and let R be a transversal of the cosets G/H. The induced representation combines $|R|$ many copies of W, as stated in the definition:

$$V = \bigoplus_{r \in R} \rho_r W.$$

Therefore, if ρ_s takes some w_i "outside" its original copy, the diagonal term must be zero. Now, let $u \in G$ be given. For a given space $\rho_r W$, in order that the space stays inside itself, we need $\rho_u \rho_r W = \rho_r W$. But this is true if and only if ur and r are in the same left coset of H, i.e. $r^{-1}ur \in H$. We hence only need to consider those spaces. Suppose $\rho_r W$ is such a space, then its basis is $\{rw_1, \cdots, rw_n\}$. Since $r^{-1}ur \in H$, suppose $ur = rh$, then the new basis of this space is given by $\{urw_1, \cdots, urw_n\} = \{r(hw_1), \cdots, r(hw_n)\}$. Since $h \in H$, we already know what it does to w_j's:

$$hw_j = \sum_{i=1}^{n} M_{ij} w_i \quad \Rightarrow \quad r(hw_j) = \sum_{i=1}^{n} M_{ij}(rw_i),$$

where M is the matrix of the transformation ρ_h on W. Hence, the action of ρ_u on $\rho_r W$ is given by the same matrix as the action of $\rho_h (= \theta_h)$ on W. Thus, they share the same trace, and we simply need to sum up the contribution of all such spaces $\rho_r W$ that contribute to the character:

$$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(h) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta\left(r^{-1}ur\right).$$

4 More on Induction

Recall the definition of induced representation. Let $H \leq G$, and let R be a system of representatives of G/H. Then (ρ, V) is said to be induced by (θ, W) if

$$V = \bigoplus_{r \in R} \rho_r W \quad \Leftrightarrow \quad V = \bigoplus_{\sigma \in G/H} W_\sigma$$

This property can be reformulated in the following way: Let

$$W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

be the $\mathbb{C}[G]$-module obtained from W by scalar extension from $\mathbb{C}[H]$ to $\mathbb{C}[G]$. Note that W' has basis $\{r \otimes w_i \mid r \in R, w_i \in \beta(W)\}$, where $\beta(W)$ is a basis of W. Since $\mathbb{C}[H]$ are like the "scalars", a choice of coset representatives spans all of $\mathbb{C}[G]$ over $\mathbb{C}[H]$. Thus, the action of $\mathbb{C}[G]$ on W' is defined by

$$g \cdot (r \otimes w) = gr \otimes w,$$

where $r \in R$ and $g \in G$.

In this sense, the following statements become evident:

1. If V is induced by W and if E is a $\mathbb{C}[G]$-module, we have a canonical isomorphism

$$\mathrm{Hom}^H(W, E) \cong \mathrm{Hom}^G(V, E)$$

2. Induction is transitive: if G is a subgroup of a group K, we have

$$\mathrm{Ind}_G^K \left(\mathrm{Ind}_H^G(W) \right) \cong \mathrm{Ind}_H^K(W).$$

   This can be seen directly, or by using the associativity of the tensor product.