06/13/22 Notes

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1 Tensor Square Decomposition

Let V be a vector space. Define an endomorphism T of $V \otimes V$ as follows:

$$\begin{aligned} T : V \otimes V &\longrightarrow V \otimes V \\ v \otimes w &\longmapsto w \otimes v. \end{aligned}$$

It is an involution (it is its own inverse), and so is an automorphism (self-isomorphism) of $V \otimes V$. Define two subsets of the second tensor power of V,

$$\begin{aligned} \mathrm{Sym}^2(V) &:= \{v \in V \otimes V \mid T(v) = v\} \\ \mathrm{Alt}^2(V) &:= \{v \in V \otimes V \mid T(v) = -v\} \end{aligned}$$

These are the symmetric square and the alternating square of $V, V \wedge V$, respectively. The symmetric and alternating squares are also known as the symmetric part and antisymmetric part of the tensor product.

Since each tensor can be written as

$$x \otimes y = \frac{x \otimes y + y \otimes x}{2} + \frac{x \otimes y - y \otimes x}{2},$$

and also $\mathrm{Sym}^2(V) \cap \mathrm{Alt}^2(V) = \{0\}$, we have that the second tensor power of a linear representation V of a group G decomposes as the direct sum of the symmetric and alternating squares:

$$V^{\otimes 2} \cong \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$$

as representations. In particular, both are **subrepresenations** of the second tensor power. In the language of modules over the group ring, the symmetric and alternating squares are $\mathbb{C}[G]$-submodules of $V \otimes V$.

If V has a basis $\{e_1, e_2, \ldots, e_n\}$, then the symmetric square has a basis $\{e_i \otimes e_j + e_j \otimes e_i \mid 1 \le i \le j \le n\}$ and the alternating square has a basis $\{e_i \otimes e_j - e_j \otimes e_i \mid 1 \le i < j \le n\}$. Accordingly,

$$\begin{aligned} \dim \mathrm{Sym}^2(V) &= \frac{n(n+1)}{2}, \\ \dim \mathrm{Alt}^2(V) &= \frac{n(n-1)}{2}. \end{aligned}$$

More generally, one can define the $\mathrm{Sym}^k(V) \subset V^{\otimes k}$ as the subspace consisting of all vectors in $V^{\otimes k}$ that are invariant under all permutations of k symbols in S_k. Thus, it's easy to see that

$$\dim \mathrm{Sym}^k(V) = \binom{n+k-1}{k}.$$

Then define

$$\mathrm{Sym}(V) := \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V).$$

Thus, $\mathrm{Sym}(V)$ is a vector subspace of $T(V)$. Moreover $\mathrm{Sym}(V)$ is isomorphic as vector space to $S(V)$ over characteristic zero fields. However, $S(V)$ has a product structure and thus this isomorphism is not an algebra isomorphism. Let π_k be the restriction to $\mathrm{Sym}^k(V)$ of the canonical surjection $T^k(V) \to S^k(V)$. If $k!$ has an inverse in the ground field (or ring), then π_k is an isomorphism. This is always the case with a ground field of characteristic zero. Moreover, the inverse of this isomorphism is given by

$$\pi_k^{-1}(v_{i_1} \cdots v_{i_k}) = \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(i_1)} \otimes \dots v_{\sigma(i_k)}.$$

In summary, over a field of characteristic zero, the symmetric tensors and the symmetric algebra form two isomorphic graded vector spaces. They can thus be identified as far as only the vector space structure is concerned, but they cannot be identified as soon as products are involved.

2 Extension of Scalars and Induced Representation

2.1 Basic Ideas

Let R, S be rings. Suppose that M is a right R-module, and we have a ring homomorphism $f : R \to S$. It's easy to turn S-modules into R-modules using the homomorphism. Now we consider how to turn M into an S-module. Because of the homomorphism f, we may regard S as a left R-module. (In fact, S is a module over itself, so it's an (R, S)-module because the two actions commute). Therefore, we form the tensor product:

$$M^S = M \otimes_R S.$$

The (right) action of S on M^S is given by:

$$(m \otimes s) * s' = m \otimes (ss').$$

It's easy to check the module axioms for M^S. Associativity often causes confusion, so we check that here:

$$(m \otimes s) * s_1 s_2 = m \otimes s s_1 s_2 = (m \otimes s s_1) * s_2 = ((m \otimes s) * s_1) * s_2.$$

In summary, the extension of scalars is the tensor product of an R-module with an (R, S)-bimodule, which yields an S-module. We could also change our assumption to change the directions of all actions. If we do that, then M is a left R-module, while S is an (S, R)-bimodule, and $S \otimes_R M$ is a left S module.

2.2 Application to Induced Representation

Suppose we have groups $H \leq G$ and an $\mathbb{C}[H]$-module W. We wish to find a representation of G that is naturally induced by W. In this case, $R = \mathbb{C}[H]$ and $S = \mathbb{C}[G]$, and therefore, using the extension of scalars method discussed above, we define the induced representation to be

$$\mathrm{Ind}_H^G(W) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

From the categorical perspective, induction is a functor from $\mathbb{C}[H]$-modules to $\mathbb{C}[G]$-modules, while restriction is the opposite. This will be discussed more in the next section.

3 More on Category Theory

First, recall the definition of a functor:

Definition 3.1 (Functor)**.** A functor is a mapping between categories $\mathcal{C}$ and $\mathcal{D}$ which sends an object X in $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$ and sends a morphism $F : X \to Y$ to a morphism $F(f) : F(X) \to F(Y)$, satisfying the following properties:

1. For every object X, we have: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

2. For all morphisms $f : X \to Y$ and $g : Y \to Z$, we have: $F(g \circ f) = F(g) \circ F(f)$

Example 3.2. For any category $\mathcal{C}$ there is the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that sends each object and morphism to itself.

Example 3.3. Let $\mathbf{Mod}_R$ be the category of modules over a ring R.
Then, $\mathrm{Ind}_H^G : \mathbf{Mod}_{\mathbb{C}[H]} \to \mathbf{Mod}_{\mathbb{C}[G]}$ and $\mathrm{Res}_H^G : \mathbf{Mod}_{\mathbb{C}[G]} \to \mathbf{Mod}_{\mathbb{C}[H]}$ are both functors.

Here, restriction acts trivially on equivariant maps since every G-equivariant maps is also H-equivariant. Induction acts on an equivariant map $f : V \to W$ by: $(\mathrm{Ind}_H^G f) : x \otimes v \mapsto x \otimes f(v)$

Proof. Restriction does not change the underlying module, so $\mathrm{Res}_H^G(\mathrm{id}_V) = \mathrm{id}_V = \mathrm{id}_{\mathrm{Res}_H^G V}$, and it satisfies the identity condition. And we have by definition: $(\mathrm{Ind}_H^G \mathrm{id}_V) : x \otimes v \mapsto x \otimes v$ so induction satisfies the identity condition.
Restriction trivially preserves composition. For induction, we have:

$$(\mathrm{Ind}_H^G(g \circ f))(x \otimes v) = x \otimes g(f(v)) = (\mathrm{Ind}_H^G g \circ \mathrm{Ind}_H^G f)(x \otimes v)$$

Therefore, Ind_H^G and Res_H^G are both functors. $\square$

We define the following notion of a natural transformation as an equivalence between functors.

Definition 3.4 (Natural Transformation)**.** A natural transformation $\eta : F \to G$ between functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ maps each object $X \in \mathcal{C}$ to a morphism $\eta_X : F(X) \to G(X)$ such that for any morphism $f : X \to Y$ in $\mathcal{C}$, we have: $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

This can be summarized with the following commutative diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

A natural isomorphism is a natural transformation where each η_X is an isomorphism in $\mathcal{D}$.

Definition 3.5 (Isomorphic Categories)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there is are two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{Id}_{\mathcal{D}}$ and $GF = \mathrm{Id}_{\mathcal{C}}$.

The notion of isomorphic categories tends to be too strong for most applications, so it is more common to use an *equivalence* of categories, where we replace the equality of the two functors with natural isomorphisms.

Definition 3.6 (Equivalence of Categories)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there are two functors $F : \mathcal{C} \to \mathcal{D}$ $G : \mathcal{D} \to \mathcal{C}$ such that both FG and $\mathrm{Id}_{\mathcal{D}}$ and also GF and $\mathrm{Id}_{\mathcal{C}}$ are naturally isomorphic.

It is easy to see that the equivalence of categories acts like an equivalence relation. This is equivalent to an alternate definition that there is an essentially surjective functor $F : \mathcal{C} \to \mathcal{D}$ such that the induced map $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$ is a bijection.

The following is some background for the definition of an adjunction between categories.

Definition 3.7 (Opposite Category)**.** The opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is the category consisting of the same objects as $\mathcal{C}$, but with the hom-classes reversed, so that $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Each identity morphism stays the same and composition is given by: $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$. Opposite categories are related to the notion of a contravariant functor:

Definition 3.8 (Contravariant Functor)**.** A *contravariant* functor is a mapping between categories $\mathcal{C}$ and $\mathcal{D}$ which sends an object X in $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$ and sends a morphism $F : X \to Y$ to a morphism $F(f) : F(Y) \to F(X)$, satisfying the following properties:

1. For every object X, we have: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
2. For all morphisms $f : X \to Y$ and $g : Y \to Z$, we have: $F(g \circ f) = F(f) \circ F(g)$.

This is an unfortunate naming convention because contravariant functors are NOT actually functors. However, they can be uniquely identified either with a functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ or a functor $H : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ since $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and $\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(F(X), F(Y)) = \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$.

Definition 3.9 (Product Category)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, the product category $\mathcal{C} \times \mathcal{D}$ is the category whose objects are of the form (C, D) for objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$ and whose morphisms $\mathrm{Hom}((C_1, D_1), (C_2, D_2))$ are of the form (f, g) for morphisms $f : C_1 \to C_2$ and $g : D_1 \to D_2$.

Composition and identities are as one would expect:

$$(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2)$$

$$\mathrm{id}_{(X,Y)} = (\mathrm{id}_X, \mathrm{id}_Y)$$

Example 3.10. If $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ are functors between (locally small) categories $\mathcal{C}$ and $\mathcal{D}$, then $\mathrm{Hom}_{\mathcal{C}}(F-, -)$ and $\mathrm{Hom}_{\mathcal{D}}(-, G-)$ are both functors $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. Where they act on a morphism (f, g) by $\mathrm{Hom}_{\mathcal{C}}(Ff, g) : h \mapsto g \circ h \circ F(f)$ and $\mathrm{Hom}_{\mathcal{D}}(f, Gg) : h \mapsto G(g) \circ h \circ f$.

Proof. First, for any object $(D, C) \in \mathcal{D}^{\mathrm{op}} \times \mathcal{C}$, the identity morphism is $(\mathrm{id}_D, \mathrm{id}_C)$. So, we have:

$$\mathrm{Hom}_{\mathcal{C}}(F\,\mathrm{id}_D, \mathrm{id}_C)(h) = \mathrm{id}_C \circ h \circ F(\mathrm{id}_D) = \mathrm{id}_C \circ h \circ \mathrm{id}_{F(D)} = h$$
$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_D, G\,\mathrm{id}_C)(h) = G(\mathrm{id}_C) \circ h \circ \mathrm{id}_D = \mathrm{id}_{G(C)} \circ h \circ \mathrm{id}_D = h$$

Now, for any two morphisms $(f_1, f_2) : (X_1, X_2) \to (Y_1, Y_2)$ and $(g_1, g_2) : (Y_1, Y_2) \to (Z_1, Z_2)$, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(F(g_1 \circ f_1), g_2 \circ f_2)(h) &= g_2 \circ f_2 \circ h \circ F(g_1 \circ f_1) \\
&= g_2 \circ f_2 \circ h \circ F(f_1) \circ F(g_1) \\
&= \mathrm{Hom}_{\mathcal{C}}(Fg_1, g_2)(\mathrm{Hom}_{\mathcal{C}}(Ff_1, f_2)(h)) \\
\mathrm{Hom}_{\mathcal{D}}(g_1 \circ f_1, G(g_2 \circ f_2))(h) &= G(g_2 \circ f_2) \circ h \circ g_1 \circ f_1 \\
&= f_2 \circ g_2 \circ h \circ g_1 \circ f_1 \\
&= \mathrm{Hom}_{\mathcal{D}}(f_1, Gf_2)(\mathrm{Hom}_{\mathcal{D}}(f_1, Gf_2)(h))
\end{aligned}$$

Note that the composition in $\mathcal{D}^{\mathrm{op}}$ leads to a reversed order of composition in some of the above. □

Definition 3.11 (Adjunction)**.** An adjunction between the (locally small) categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{C}}(F-, -)$ and $\mathrm{Hom}_{\mathcal{D}}(-, G-)$ are naturally isomorphic.

In this case, F is called the left-adjoint and G is called the right-adjoint.
Now, for our canonical example of an adjunction:

Example 3.12. For groups G and H with $H \leq G$ and finite index, Ind_H^G and Res_H^G form an adjunction.

That $\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G -, -)$ and $\mathrm{Hom}_{\mathbb{C}[H]}(-, \mathrm{Res}_H^G -)$ are naturally isomorphic can be seen as a categorification of Frobenius Reciprocity, which is: if G and H are finite groups and $H \leq G$ and we have an H-character χ and a G-character φ, then:

$$\left(\mathrm{Ind}_H^G \chi \,\middle|\, \varphi\right)_G = \left(\chi \,\middle|\, \mathrm{Res}_H^G \varphi\right)_H$$