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1 Graded Rings, Modules, and Algebras

Definition 1.1. A graded ring R is a ring $R = \bigoplus_{n\geq 0} R_n$, where the R_n are abelian groups under the ring's addition operation, such that the ring's multiplication operation is a bilinear map $\cdot : R_m \times R_n \to R_{m+n}, \forall n, m \in \mathbb{Z}_{\geq 0}$

Definition 1.2. A graded module M over a graded ring R is a left R-module $M = \bigoplus_{n\geq 0} M_n$ where the M_n are abelian and such that scalar multiplication is a bilinear map $\cdot : R_n \times M_m \to M_{n+m}, \forall n, m \in \mathbb{Z}_{\geq 0}$.

Remark 1.3. Since associativity of the ring's multiplication operation was not used in the above definitions, these definitions are still valid for non-associative algebras over a field. Additionally, M being a left R-module is an arbitrary choice, and one could define what it means for right R-modules to be graded in a similar way.

Remark 1.4. R_0 is a subring of R.

Definition 1.5. A homomorphism of graded modules, is a homomorphism of R-modules $f : N \to M$ such that $\forall i \in \mathbb{Z}_{\geq 0}, f(N_i) \subseteq M_i$.

Definition 1.6. Let M be a graded R-module, and N be submodule of M. Then we say N is a graded submodule of M if the inclusion map $N \hookrightarrow M$ is a homomorphism of graded modules, or equivalently, if $\forall i \in \mathbb{Z}_{\geq 0}, N_i \subseteq M_i$

Proposition 1.7. *Let N be a (not necessarily graded) submodule of a graded R-module, M. Then $\bigoplus_{n\geq 1} N \cap M_i$ is a graded R-submodule of M.*

Proof. This can shown in a straightforward way by simply unraveling the definitions and checking that the conditions for being a graded R-submodule of M are satisfied. □

Example 1.8. Any graded ring R is graded as an R-module.

Example 1.9. $\mathrm{T}(V)$ is a graded associative $\mathbb{K}$-algebra, where V is a vector space over $\mathbb{K}$. That is, it is an associative algebra over $\mathbb{K}$ that is graded when considered as a ring.

Definition 1.10. A two-sided ideal I of a graded ring R is a homogeneous ideal if it is a graded R-submodule of R. Equivalently, I is a homogeneous ideal of R if it can be generated by elements of $\bigcup_{n\geq 0} R_n \cap I$.

Proposition 1.11. *Let R be a graded ring, and let I be homogeneous ideal of R. Then $R/I = \bigoplus_{n\geq 0} R_n/(R_n \cap I)$ is a graded ring.*

Proof. Let $I_i := R_i \cap I$, $(R/I)_i := R_i/I_i$. Then we see that since $I_i \subseteq R_i$ are abelian groups, $(R/I)_i$ must be as well. Furthermore, $\forall r_i \in R_i, r_j \in R_j$, we have $(r_i + I_i) \cdot (r_j + I_j) = (r_i \cdot r_j + r_i I_j + I_i r_j + I_i I_j) \subseteq r_i \cdot r_j + I_{i+j} \in (R/I)_{i+j}$. The equality between R/I and the direct sum of the $(R/I)_n$ follows immediately from the definition of the quotient of a ring by an ideal and the fact that R and I are both equal to the direct sum of their graded pieces, R_n and $R_n \cap I$ respectively. $\square$

Definition 1.12. Let V be a graded $\mathbb{K}$-algebra such that $\forall i \in \mathbb{Z}_{n\geq 0}, \dim_{\mathbb{K}} V_n$ is finite. Then the Hilbert-Poincare series is defined as the formal power series $\mathrm{HP}(V) := \sum_{n\geq 0} (\dim_{\mathbb{K}} V_n) q^n$.

Example 1.13. Let V be an n-dimensional vector space over $\mathbb{K}$. Then since $V^{\otimes k}$ has dimension n^k, we have $\mathrm{HP}(\mathrm{T}(V)) = \frac{1}{1-nq}$.

2 The Symmetric Algebra, S(V)

Definition 2.1. Let $\mathrm{T}(V)$ be the tensor algebra of a finite dimensional vector space V over a field $\mathbb{K}$, and let I be the two-sided ideal of $\mathrm{T}(V)$ generated by $\{x \otimes y - y \otimes x : x, y \in V\}$. Then we say the symmetric algebra of V is $\mathrm{S}(V) := \mathrm{T}(V)/I$.

Remark 2.2. $\mathrm{S}(V)$ can be thought of as a coordinate free polynomial ring, since if we fix a basis, $v_1, ..., v_n$, we see $\mathrm{S}(V)$ is isomorphic to the polynomial ring $\mathbb{K}[v_1, ..., v_n]$.

Proposition 2.3. $\mathrm{S}(V)$ *is a graded* $\mathbb{K}$-*Algebra.*

Proof. From Proposition 1.11, it suffices to check that I is a homogeneous ideal of the tensor algebra $\mathrm{T}(V) := \bigoplus_{n\geq 0} V^{\otimes n}$. Indeed, I is generated by $\{x \otimes y - y \otimes x : x, y \in V\} \subseteq V^{\otimes 2}$, which concludes the proof. $\square$

Remark 2.4. The elements of $\mathrm{S}(V^*)$ can be thought of as polynomial functions on the vector space V, since one can fix a basis $x_1, ...x_n$ of V^*, express a given element p_V of $\mathrm{S}(V*)$ as a polynomial p in the x_i, and define $p_V(v) := p(x_1(v), ..., x_n(v))$.

Remark 2.5. From Remark 2.2 and Proposition 2.3, we see that if V is an n-dimensional vector space over $\mathbb{K}$ the dimension of $\mathrm{S}^k(V)$ is the number of degree k monomials in n variables, which from the method of stars and bars from enumerative combinatorics, can be shown to be $\binom{n+k-1}{k-1}$.

Combining this with the negative-binomial theorem, one obtains

Corollary 2.6. $\mathrm{HP}(\mathrm{S}(V)) = \frac{1}{(1-q)^n}$

3 The Exterior Algebra

Definition 3.1. Let $\mathrm{T}(V)$ be the tensor algebra of a finite dimensional vector space V over a field $\mathbb{K}$, and I be the two-sided ideal of $\mathrm{T}(V)$ generated by the set $\{x \otimes x | x \in V\}$. The exterior algebra of V is $\bigwedge(V) := \mathrm{T}(V)/I$. For $v, w \in T(V)$, we call $v \wedge w = [v \otimes w]$ the wedge product of v and w. Note that the wedge product is anti-commutative.

$$\begin{aligned} 0 &= (v + w) \wedge (v + w) \\ &= v \wedge v + v \wedge w + w \wedge v + w \wedge w \\ &= v \wedge w + w \wedge v \end{aligned}$$

Corollary 3.2 (Sign Property). *From this, one can see that* $\forall \pi \in S_k, v_1, ..., v_k \in V$*, we have* $v_{\pi(1)} \wedge ... \wedge v_{\pi(n)} = \operatorname{sgn}(\pi) v_1 \wedge ... \wedge v_k$*, where* $\operatorname{sgn}(\pi)$ *is the sign of the permutation* π*.*

Proposition 3.3. $\bigwedge(V)$ *is a graded algebra.*

Proof. The ideal I as defined above is generated by $\{x \otimes x : x \in V\} \subseteq V^{\otimes 2}$, so it is a homogeneous ideal of $\mathrm{T}(V)$, and so the claim follows from Proposition 1.11 $\square$

Proposition 3.4. *If* V *is a n-dimensional vector space, the dimension of* $\bigwedge^k(V)$ *is* $\binom{n}{k}$*.*

Proof. Fix a basis $e_1, ..., e_n$ of V. Then one can use linearity and the sign property to show that any wedge product of k vectors in V decomposes into a linear combination of wedge products of k of the e_i. Furthermore this spanning set for $\bigwedge^k(V)$ is linearly independent, since if $e_{i_1}, ..., e_{i_k}$ were equal to a linear combination of the other basis elements in $\bigwedge^k(V)$, then taking the wedge product on both sides with the element of $\bigwedge^{n-k}(V)$ given by the wedge product of all the other basis vectors, we would have the equality $e_1 \wedge ... \wedge e_n = 0$ in $\bigwedge(V)$, which is false. We see that each of these basis elements can can be uniquely specified by a subset of k of the e_i, so the size of basis is $\binom{n}{k}$. $\square$

Corollary 3.5. *The Hilbert-Poincare series of* $\bigwedge(V)$ *is* $(1+q)^n$

Proof. $HP(\bigwedge(V)) = \sum_{i=0}^{n} \binom{n}{m} q^n$. Then use the binomial theorem. $\square$

Remark 3.6. From the method used in the proof above to show that the wedge products of k of the basis vectors $e_1, ..., e_n$ are linearly independent, one can also show that after fixing a basis for V, one there is a canonical isomorphism between $\bigwedge^k(V)^*$ and $\bigwedge^{n-k}(V)$.

Remark 3.7. If V is a n-dimensional $\mathbb{K}$-vector space, and $x \in \bigwedge(V)$, then x looks like linear combinations of $v = v_1 \wedge v_2 \wedge \ldots \wedge v_k$ for $v_i \in V$. If $k > n$ then $v = 0$.

Proof. If $k > n$ then the set $\{v_1, \ldots, v_k\}$ is linearly dependent. In particular $v_1 = \sum_{i=2}^{k} a_i v_i$ for $a_i \in \mathbb{K}$. Then $v = (a_2 v_2 + \ldots a_k v_k) \wedge v_2 \wedge \ldots \wedge v_k = 0$ $\square$

Now suppose $T : V \to V$ is a linear map. Then T induces a map $T_k : V^{\otimes k} \to V^{\otimes k}$ where $T_k(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = T(v_1) \otimes T(v_2) \otimes \ldots \otimes T(v_k)$. Taking the direct sum over all k, we see that T induces a graded K-algebra endomorphism of $T(V)$. Furthermore, if I is homogeneous ideal of $\mathrm{T}(V)$ that is closed under the action of this endomorphism of $\mathrm{T}(V)$, then this passes to an induced endomorphism $T'_k : V^{\otimes n}/I_k \to V^{\otimes n}/I_k$.

Example 3.8. If we take I to be the ideal we used to define the exterior algebra, and additionally take $k = n$, then $\bigwedge^k(V)$ is one dimensional. So $T'_n(v) = dv$ for some scalar d. We call d the determinant of T. This gives us a coordinate-free description of the determinant since at no point did we have to discuss a particular basis.

Example 3.9. From Remark 3.5 above, it also follows that any linear map $T : V \to V$ induces a map $T' : \mathrm{S}^k(V) \to \mathrm{S}^k(V)$, which can be seen as a coordinate-free way of defining the polynomial representations $\mathrm{Sym}^k(\mathbb{K}^n)$.

Remark 3.10. As it turns out, if V is an inner product space of dimension n over a field K, then the inner product on V induces an inner product on $\bigwedge^k(V)$. This allows one to prove higher dimensional generalizations of the pythagorean theorem, such as De Gua's Theorem.