05/23/22 Notes

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- Next meeting time: 10am on Tuesday at AH347

1 Representations of $(\mathbb{Z}, +)$

Proposition 1.1. *Let $\varphi : \mathbb{Z} \to \mathrm{GL}_n(\mathbb{C})$ be a representation of $(\mathbb{Z}, +)$. Then φ is equivalent to the representation ρ defined by $\rho_n = J_{\varphi_1}^n$, where J_M denotes the Jordan canonical form of a matrix M, which is unique up to reordering the Jordan blocks.*

Proof. Let A denote φ_1, the image of 1 under φ. Then φ must be $\varphi_k = A^k$ because 1 generates $\mathbb{Z}$. Note that φ is indeed a representation because $\varphi_{n+m} = A^{n+m} = A^n A^m = \varphi_n \varphi_m$, assuming A is invertible. Next, observe that because J_A is similar to A, there exists $T \in \mathrm{GL}_n(\mathbb{C})$ such that $TJ_AT^{-1} = A$, and so

$$(TJ_AT^{-1})^n = T{J_A}^nT^{-1} = A^n \quad \Rightarrow \quad T\rho_nT^{-1} = \varphi_n \quad \forall n \in \mathbb{Z},$$

which proves that $\varphi \sim \rho$. □

Remark. In fact, J_M can be substituted by any matrix similar to M, but the Jordan canonical form is, in some sense, the "most diagonal" one.

2 Dual Representation

Definition 2.1. For a vector space V over F, we define the dual space of V to be the vector space $\mathrm{Hom}(V, F)$, denoted by V^*.

Theorem 2.2. *Let V and W be finite-dimensional vector spaces over F with ordered bases β and γ, respectively. For any linear transformation $L : V \to W$, the mapping $L^\mathsf{T} : W^* \to V^*$ defined by $L^\mathsf{T}(g) = gL$ for all $g \in W^*$ is a linear transformation with the property that $\left[L^\mathsf{T}\right]_{\gamma^*}^{\beta^*} = \left([L]_\beta^\gamma\right)^\mathsf{T}$.*

Theorem 2.3. *Let $\varphi : G \to \mathrm{GL}(V)$ be a representation of G. Then there is a corresponding representation of G, defined by: $\varphi_g^* = (\varphi_{g^{-1}})^\mathsf{T}$.*

Proof. For the sake of convenience, we're only dealing with matrices here. Theorem 2.2 provides a better way to understand the "transpose" of a linear map. Now, it suffices to show that $\varphi_{gh}^* = \varphi_g^*\varphi_h^*$. This is so because

$$\varphi_{gh}^* = (\varphi_{(gh)^{-1}})^\mathsf{T} = (\varphi_{h^{-1}g^{-1}})^\mathsf{T} = (\varphi_{g^{-1}})^\mathsf{T}(\varphi_{h^{-1}})^\mathsf{T} = \varphi_g^*\varphi_h^*,$$

which concludes the proof. □

3 Character Theory

In this section we will only consider finite groups.

Definition 3.1 (Character)**.** For a group G, the character of a group representation $f : G \to \mathrm{GL}(V)$ is defined as $\chi_f : G \to \mathbb{C}$ where $\chi_f(g) = \mathrm{Tr}(f_g)$.

Definition 3.2. The kernel of a character χ_f is the set $\{g \in G | \chi_f(g) = \chi_f(1)\}$.

Definition 3.3. The degree of a character χ_f is $\deg(f)$.

Definition 3.4. A character χ_f is irreducible if f is irreducible.

We now list some facts about the character of a finite group.

Theorem 3.5. $\chi_f(1) = \deg(f)$

Proof. By definition, $\chi_f(1) = \mathrm{Tr}(f(1)) = \mathrm{Tr}(I_V) = \dim(V) = \deg(f)$ □

Theorem 3.6 (Weyl's Unitarian Trick)**.** *All linear representations of a finite group are equivalent to unitary representations.*

Proof. We want to find a change of basis and an inner product such that our linear maps become unitary. Over $V = \mathbb{C}^n$ we have the standard inner product or the dot product. This product is bad for representation theory though since $f(g)$ is not an isometry. i.e $\langle v, w\rangle \neq \langle g \cdot v, g \cdot w\rangle$. Our fix is to define a new inner product

$$\langle v, w\rangle' := \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w\rangle$$

We can check that this is a valid inner product and is G-invariant. i.e it satisfies conjugate symmetry, linearity in the first argument, positive definiteness and definition 3.3.

$$\begin{aligned}
\langle v, w\rangle' &= \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w\rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\langle g \cdot w, g \cdot v\rangle} \\
&= \overline{\langle w, v\rangle'}
\end{aligned}$$

$$\begin{aligned}
\langle av + bu, w\rangle' &= \frac{1}{|G|} \sum_{g \in G} \langle g \cdot (av + bu), g \cdot w\rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \langle g \cdot (av), g \cdot w\rangle + \langle g \cdot (bu), g \cdot w\rangle \\
&= \frac{a}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w\rangle + \frac{b}{|G|} \sum_{g \in G} \langle g \cdot u, g \cdot w\rangle \\
&= a\langle v, w\rangle' + b\langle u, w\rangle'
\end{aligned}$$

$$\langle v, v\rangle' = \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot v\rangle \geq 0 \text{ with equality } \iff g \cdot v = 0, \forall g \in G \iff v = 0$$

$$\begin{aligned}\langle h\cdot v, h\cdot w\rangle' &= \frac{1}{|G|}\sum_{g\in G}\langle hg\cdot v, hg\cdot w\rangle \\ &= \frac{1}{|G|}\sum_{g'\in G}\langle g'\cdot v, g'\cdot w\rangle \text{ (note } hg \text{ simply reorders the elements since G is a finite group)} \\ &= \langle v, w\rangle'\end{aligned}$$

Now that we have a suitable inner product, we can perform the Gram-Schmidt process on any basis of V to produce an orthonormal basis. Let $\beta = \{e_1, e_2, \ldots, e_n\}$ be the standard basis and call this o.n.b $\gamma = \{v_1, v_2, \ldots, v_n\}$. Now we can let $Q = [I_V]_\gamma^\beta$, then the claim is that $Q^{-1}[f_g]_\beta Q$ is a unitary matrix, where f_g is written with respect to the standard basis. This is true because,

$$Q^{-1}[f_g]_\beta Q = [I_V]_\beta^\gamma [f_g]_\beta [I_V]_\gamma^\beta = [f_g]_\gamma,$$

but γ is an orthonormal basis with respect to our newly defined inner product, and f is also unitary with respect to that inner product, so $[f_g]_\gamma$ must be a unitary matrix. □

Theorem 3.7. $\chi_f(s^{-1}) = \overline{\chi_f(s)}$.

Proof. $\overline{\chi_f(s)} = \overline{\text{Tr}(f(s))}$. Using fact 2 and the fact that all unitary matrices are diagonalizable, $\overline{\text{Tr}(f(s))} = \sum_{j=1}^n \overline{\lambda_j}$. The eigenvalues of a unitary operator satisfy $|\lambda| = 1$ or equivalently $\lambda \in U(1)$.

$$\overline{\lambda_j} = a_j - b_j i = \frac{a_j - b_j i}{a^2 + b^2} = \frac{1}{a_j + b_j i} = \lambda_j^{-1}$$

. Therefore,

$$\overline{\text{Tr}(f(s))} = \sum_{j=1}^n \frac{1}{\lambda_j} = \chi_f(s^{-1})$$

□

Theorem 3.8. $\chi_f(tst^{-1}) = \chi(s)$.

Proof. By definition,

$$\chi_f\left(hgh^{-1}\right) = \text{Tr}(f(hgh^{-1})) = \text{Tr}(f(h)f(g)f(h^{-1})) = \text{Tr}(f(h^{-1})f(h)f(g)) = \text{Tr}(f(g)) = \chi_f(g),$$

where we used the fact that $\text{Tr}(AB) = \text{Tr}(BA)$. □

Definition 3.9 (Conjugacy class)**.** The conjugacy classes of a group G are the set of equivalence classes formed by the equivalence relation $a \sim b \iff \exists g \in G, a = g^{-1}bg$

Definition 3.10 (Class function)**.** A class function is a function that is constant on the conjugacy classes of G.

Fact 4 asserts that the character of a representation is a class function. In fact, the irreducible characters form an orthonormal basis of the vector space of class functions of G under the following inner product.

$$\langle \alpha, \beta\rangle = \frac{1}{|G|}\sum_{g\in G}\alpha(g)\overline{\beta(g)}$$