05/20/22 Notes

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- Next meeting time: 10am on Monday at AH347

1 Polynomial and Rational Representations

Definition 1.1 (Polynomial Representations)**.** A polynomial representation is a representation $f : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_m(\mathbb{K})$ such that each entry of $f(A)$ is some fixed polynomial in the entries of A.

Definition 1.2 (Rational Representations)**.** A rational representation f is defined in the same way, except each entry of $f(A)$ is required to be a ratio of two polynomial expressions in the entries of A.

Proposition 1.3. *Over an algebraically closed field $\mathbb{K}$, any rational representation $f : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_m(\mathbb{K})$ is of the form $\frac{g}{\mathrm{Det}^k}$, where $g : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_m(\mathbb{K})$ is a polynomial representation.*

Proof. Observe that since f must be a function, each entry of $f(X)$ must be well defined for all n^2-tuples $X_{i,j}, i, j = 1...n$ such that the matrix X whose i, j-th entry is $X_{i,j}$ has nonzero determinant. It suffices to prove that for any given entry $\frac{p(...,X_{i,j},...)}{q(...,X_{i,j},...)}$ of the rational representation, there is some non negative integer k and some other polynomial $t(..., X_{i,j}, ...)$ in n^2 variables such that $q(..., X_{i,j}, ...) \cdot t(..., X_{i,j}, ...) = \mathrm{Det}(X)^k$. This is because to obtain the original claim, we can use $\frac{p}{q} = \frac{p \cdot t}{\mathrm{Det}^k}$, and factor $\frac{1}{\mathrm{Det}^{k_0}}$ out of the matrix, where k_0 is the maximum value of k over all entries of $f(X)$. Now, using our observation that $q(..., A_{i,j}, ...)$ must be nonzero when the determinant of $A \in M_{n \times n}(\mathbb{K})$ is nonzero, we see that $\mathrm{Det}(X)$ is contained in the ideal of all polynomials in n^2 variables that vanish (are 0) at all points of $\mathbb{K}^{n^2}$ where q vanishes. So by applying the Nullstellensatz to the ideal given by $\{q \cdot a : a \in \mathbb{K}[..., X_{i,j}, ...]\}$, we see that there exists some $t \in \mathbb{K}[..., X_{i,j}, ...]$ and some integer $k \geq 0$ such that $q(..., X_{i,j}, ...) \cdot t(..., X_{i,j}, ...) = \mathrm{Det}(X)^k$, and as outlined above, the result follows. $\square$

Now, we give an example of a polynomial representation

Example 1.4 ($\mathrm{Sym}^2(\mathbb{K}^2)$)**.** Let $G = \mathrm{GL}_2(\mathbb{K})$, and let V be the $\mathbb{K}$-vector space of homogeneous polynomials of degree 2 in 2 variables with coefficients in $\mathbb{K}$ (which has a basis given by x^2, xy, y^2). Let $g \in G$ act on V by a linear change of coordinates $x \to$ To be more precise, given $g \in \mathrm{GL}(\mathbb{K}^2)$, $g \cdot p(x, y) := p(g_{1,1}x + g_{1,2}y, g_{2,1}x + g_{2,2}y)$. In the basis x^2, xy, y^2, the corresponding map $f : \mathrm{GL}_2(\mathbb{K}) \to \mathrm{GL}_3(\mathbb{K})$ is given by

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

2 Group Actions and Permutation Representations

Definition 2.1 (Group Action)**.** Let G be a group and X be a set. A group action of G on X is a map $G \times X \to X$, denoted by $x \mapsto g \cdot x$, such that the following conditions hold

- (Identity) $e_G \cdot x = x$ for all $x \in X$, where e_G is the identity element of G.
- (Associativity) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$.

Remark. Note that the maps $g \cdot (-) : X \to X$ associated to each $g \in G$ by the group action are all bijective, because the inverse of $g \cdot (-)$ is given by the map $g^{-1} \cdot (-)$. So for a finite set X, the group action associated to an element $g \in G$ is a permutation of the elements of X.

Definition 2.2 (Permutation Representation)**.** Let G be a group that acts on a set X, and let $\mathbb{K}$ be a field. Consider the free vector space V consisting of all finite formal linear combinations of elements of X over $\mathbb{K}$. This can be made into a G-module by defining $g \cdot (1 \cdot x) = 1 \cdot (g \cdot x)$ and extending the action of G on elements of V of the form $1 \cdot x$ linearly to all elements of V. From the previous remark, one can see that this is a group homomorphism $G \to GL(V)$, which is a representation of G.

Definition 2.3 (Free Actions)**.** A group action G on X is a free action if $g \cdot x = x$ implies g is the identity of G.

Definition 2.4 (Transitive Actions)**.** A group action G on X is transitive if X is non-empty and if for each pair x, y in X there exists a g in G such that $g \cdot x = y$.

Definition 2.5 (Orbit)**.** The orbit of $x \in X$ is $\{g \cdot x \mid g \in X\}$. Intuitively, this is the set of all elements in X that can be reached by x.

Definition 2.6 (Stabilizer)**.** For every x in X, the stabilizer subgroup of G with respect to x is the set of all elements in G that fix x, that is, $\{g \in G \mid g \cdot x = x\}$. Note that the stabilizer subgroup is indeed a subgroup of G.

Definition 2.7 (Regular Representation)**.** Let G be a finite group of order n and $\mathbb{K}$ be a field. Let the group ring (algebra) $\mathbb{K}[G]$ be defined as the $\mathbb{K}$-vector space of all formal linear combinations of $g_i \in G$.

$$\mathbb{K}[G] := \left\{ \sum_{i=1}^{n} a_i g_i \mid a_i \in \mathbb{K}, g_i \in G \right\}$$

The multiplication operation of $\mathbb{K}[G]$, $\cdot$, is defined to be the unique bilinear map $\mathbb{K}[G] \times \mathbb{K}[G] \to \mathbb{K}[G]$ such that for all $g, h \in G$, $g \cdot h = gh$.

- This multiplication operation turns $\mathbb{K}[G]$ into a $\mathbb{K}$-algebra
- The collection $\{g : g \in G\}$ of elements of $\mathbb{K}[G]$ form a basis for $\mathbb{K}[G]$ over $\mathbb{K}$.

Since $\mathbb{K}[G]$ is a vector space, for each $g \in G$, define the map $\mathbb{K}[G] \to \mathbb{K}[G]$ by $g\left(\sum a_i g_i\right) = \sum a_i(gg_i)$. These maps are invertible linear transformations of $\mathbb{K}[G]$, so $\mathbb{K}[G]$ is a representation of G. Note that this is just a special case of permutation representation where $X = G$, and the group action is given by the group operation.

3 Building New Representations from Old Ones

There are a few ways to construct a new representation from old ones: direct sum, tensor product, and restriction.

Definition 3.1 (Restriction)**.** Let H be a subgroup of G and let $f : G \to \mathrm{GL}(V)$ be a representation of G. We can restrict the domain of f to H, denoted by $f|_H$ or $\mathrm{Res}_H^G(V)$, and it would still be a group homomorphism. Thus $f|_H$ is obviously a representation of H.

Remark. The converse is harder, namely to construct a representation of G from H.

Definition 3.2 (External Direct Sum)**.** Let W_1 and W_2 be two abelian groups. Then their direct sum is defined the same structure of the set of ordered pairs

$$W_1 \oplus W_2 := \{(w_1, w_2) \mid w_1 \in W_1, w_2 \in W_2\}$$

with addition defined coordinate-wise. When both W_1 and W_2 carry additional structure, that structure is transferred to the direct sum in a coordinate-wise manner. For two vector spaces over the same base field, scalar multiplication is given by $k(w_1, w_2) := (kw_1, kw_2)$.

Example 3.3. If W_1 and W_2 are two G-modules, then define the group action by:

$$g \cdot (w_1, w_2) := (g \cdot w_1, g \cdot w_2) \quad \forall g \in G, w_1 \in W_1, w_2 \in W_2$$

Definition 3.4. Let $f^{(1)} : G \to \mathrm{GL}(W_1)$ and $f^{(2)} : G \to \mathrm{GL}(W_2)$ be two representations of a group G. Then there exists a natural representation of G given by $(f^{(1)} \oplus f^{(2)}) : G \to \mathrm{GL}(W_1 \oplus W_2)$, and

$$\left(f^{(1)} \oplus f^{(2)}\right)_g (w_1, w_2) = \left(f_g^{(1)}(w_1), f_g^{(2)}(w_2)\right).$$

Moreover, it's easy to see that if the matrices corresponding to f_1 and f_2 are A and B, then the matrix corresponding to φ is:

$$\left(\begin{array}{c|c} A & O \\ \hline O & B \end{array}\right).$$

Remark. Example 3.3 and Definition 3.4 are equivalent because V is a G-module if and only if there exists a representation of G.

Definition 3.5 (Internal Direct Sum)**.** Given an abelian group V and two subgroups W_1 and W_2, we say V is the direct sum of W_1 and W_2 if each element of V is expressible uniquely as the sum of an element of W_1 and an element of W_2.

Remark. Note that because this way of expressing any element of V as a sum of elements of W_1 and W_2 is unique, it gives an isomorphism between V and $W_1 \oplus W_2$, which one can check is still an isomorphism even if V carries the extra structure of a ring, module, vector space, G-module, etc...

Example 3.6. Let V be a vector space, then $V = \bigoplus_{i=1}^n W_i$ if and only if:

- $V = \displaystyle\sum_{i=1}^n W_i$. That is, $\left\{\displaystyle\sum_{i=1}^n w_i \mid w_i \in W_i\right\} = V$

- $\dim V = \displaystyle\sum_{i=1}^{n} \dim W_i$.

Definition 3.7 (Projection)**.** Let V be a vector space and W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. $T : V \to V$ is called the projection on W_1 along W_2 if, for $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, we have $T(x) = x_1$. Namely, $V = \operatorname{im} T \oplus \ker T$.

Theorem 3.8 (Maschke's Theorem)**.** *Suppose G is a finite group, and V is a finite dimensional vector space over a field $\mathbb{F}$ of characteristic not dividing $|G|$. Let $f : G \to GL(V)$ be a reducible representation of G, i.e. V has a non-trivial G-invariant proper subspace W. Then there exists another G-invariant subspace $\widetilde{W}$, such that $V = W \oplus \widetilde{W}$ as G-modules.*

Proof. Let G and $W \subset V$ be given. Let W' be any subspace of V such that $V = W \oplus W'$. Note that W' may not be G-invariant. Define the projection $p : V \to W$ along W'. Define another map $\widetilde{p}$ ::

$$\widetilde{p} := \frac{1}{|G|} \sum_{t \in G} f(t) \circ p \circ f(t^{-1}).$$

Because $f(t)$ and $f(t^{-1})$ are from V to V and p is from V to W, it follows that $\widetilde{p}$ maps from V to W. Consider any $x \in W$, then $f(t^{-1})(x) \in W$ because W is G-invariant. Hence,

$$p \circ f(t^{-1})(x) = f(t^{-1})(x)$$

because p projects onto W. Therefore,

$$\widetilde{p}(x) = \frac{1}{|G|} \sum_{t \in G} f(t) f(t^{-1})(x) = x \quad \forall x \in W.$$

Thus, $\widetilde{p}$ is also a projection onto W, let $\widetilde{W} = \ker \widetilde{p}$. I claim that $f(s)$ and $\widetilde{p}$ commute for all $s \in G$. Namely, it needs to be shown that $f(s) \circ \widetilde{p} \circ f(s^{-1}) = \widetilde{p}$ for all $s \in G$. This is so because

$$f(s) \circ \widetilde{p} \circ f(s^{-1}) = \frac{1}{|G|} \sum_{t \in G} f(st) \circ \widetilde{p} \circ f((st)^{-1}) = \widetilde{p},$$

where the first equality follows because $t \mapsto st$ is a bijection from G to itself. Therefore, $f(s) \circ \widetilde{p} = \widetilde{p} \circ f(s)$ for all $s \in G$. Now, let $x \in \widetilde{W} = \ker \widetilde{p}$ and let $s \in G$, then $\widetilde{p}(x) = 0_V$, so

$$\widetilde{p} \circ f(s)(x) = f(s) \circ \widetilde{p}(x) = f(s)(0_V) = 0_V.$$

Therefore, $\widetilde{W}$ is G-invariant and $V = W \oplus \widetilde{W}$. □

Theorem 3.9. *Every representation of a finite group G is completely decomposable, i.e. is equivalent to the direct sum of finitely many irreducible representations of G (Again, provided that the characteristic of the base field does not divide the order of G).*

Proof. Let $f : G \to GL(V)$ be a representation of a finite group G. The proof is by mathematical induction on the degree of f. If $\deg(f) = 1$, then we are done because any degree one representation is irreducible. Assume the statement is true whenever $\deg(f) \leq n$, and suppose that $\deg(f) = n + 1$. If f is irreducible, then we're done again, so assume f is not irreducible. By Maschke's Theorem, V has two G-invariant subspaces V_1, V_2 and $V = V_1 \oplus V_2$, so evidently $f \sim f^{(1)} \oplus f^{(2)}$, where $f^{(1)}$ and $f^{(2)}$ are the corresponding subrepresentations of f. Since $\deg(f^{(1)}) \leq n$, we may decompose it into a direct sum of finite irreducible representations of G and do the same to $f^{(2)}$. Therefore, f is equivalent to the direct sum of finitely many irreducible representations of G. □

Definition 3.10 (Indecomposable Representation)**.** A representation $f : G \to \mathrm{GL}(V)$ is indecomposable if it is not a direct sum of irreducible subrepresentations.

Remark. An irreducible representation must be indecomposable. Theorem 3.8 just showed that a reducible representation of a finite group must be decomposable (under a mild condition on the characteristic of the base field).