05/17/22 Notes

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- Broad Topic that we're going to discuss for 2022 summer ICLUE are symmetric functions, complex (semi)-simple Lie algebra, representation.
- Tomorrow meeting time: 10am at AH347.
- Courtesy of editing/commenting: please use the \com your initial{}. For example, mine will be \comJC.

1 Group Theory

Definition 1.1 (Group)**.** A group $G = (S, \cdot)$ is a set S with a binary operation $\cdot : S \times S \to S$ satisfying:

1. (Associativity) $\forall a, b, c \in S: \ (a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. (Identity) $\exists 1 \in S, \forall a \in S : 1 \cdot a = a \cdot 1 = a$
3. (Invertibility) $\forall a \in S, \exists a^{-1} \in S : a^{-1} \cdot a = a \cdot a^{-1} = 1$

Example 1.2 (Integers)**.** The additive group of integers: $(\mathbb{Z}, +)$

Example 1.3 (Free Group)**.** Consider a set of symbols $X = \{a, b, c, \dots\}$. Let S be the set of all finite length words using the symbols $a, b, c, \dots$ and $a^{-1}, b^{-1}, c^{-1}, \dots$ where words are considered identical if they can be simplified by eliminating xx^{-1} and $x^{-1}x$ for any symbol x. A word with no symbols is also considered to be the empty word $\varnothing$. With the following, this is a group:

- $\cdot$ is concatenation (ex: $(abc^{-1}bbac^{-1}b^{-1})(bab) = abc^{-1}bbac^{-1}\underline{b^{-1}b}ab = abc^{-1}bbac^{-1}ab$.
- The identity is $\emptyset$, the empty word.
- Inverses are given by reversing the order of the word and apply the replacements $x \mapsto x^{-1}$ and $x^{-1} \mapsto x$ (ex: $(abcb)^{-1} = b^{-1}c^{-1}b^{-1}a^{-1}$)
- Associativity is trivial.

This group is typically denoted F_X or $F(X)$.

Remark: Free group is an universal object. To understand an universal object, we're going to introduce some category theory.

2 Category Theory

Definition 2.1 (Category)**.** A category $\mathcal{C}$ has objects $A, B, C, \dots$ and morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for each pair of objects A and B (if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, write $f : A \to B$) which must satisfy the properties:

1. (Identity) For any object X, there exists a morphism $\mathrm{id}_X : X \to X$ such that for any other morphisms $f : A \to X$ and $g : X \to A$ we have: $\mathrm{id}_X \circ f = f$ and $g \circ \mathrm{id}_X = g$.

2. (Associativity) For any morphisms $f : A \to B$, $g : B \to C$, and $h : C \to D$, we have: $(h \circ g) \circ f = h \circ (g \circ f)$

Definition 2.2 (Functor)**.** A functor is a mapping between categories $\mathcal{C}$ and $\mathcal{D}$ which sends an object X in $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$ and sends a morphism $F : X \to Y$ to a morphism $F(f) : F(X) \to F(Y)$, satisfying the following properties:

1. For every object X, we have: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

2. For all morphisms $f : X \to Y$ and $g : Y \to Z$, we have: $F(g \circ f) = F(g) \circ F(f)$.

In our example, $F(X)$ is a functor from the category of sets to the category of groups.

Definition 2.3 (Universal Property)**.** Consider a functor $U : \mathcal{C} \to \mathcal{D}$. Given an object X of $\mathcal{D}$, a universal morphism is a unique pair (A, u) containing an object A of $\mathcal{C}$ and a morphism $u : X \to U(A)$ such that for any object A' of $\mathcal{C}$ and any morphism $f : X \to U(A')$ there exists a unique morphism $\varphi : A \to A'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] & U(A) \arrow[d, dashed, "U(\varphi)"] & & A \arrow[d, dashed, "\exists! \varphi"] \\ & U(A') & & A' \end{tikzcd}$$

Note that Universal properties define objects uniquely up to a unique isomorphism. Therefore, one strategy to prove that two objects are isomorphic is to show that they satisfy the same universal property.

For the free group on X, the universal property is: For any group G with a function $f : X \to G$, there exists a unique homomorphism $\varphi : F_X \to G$ such the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & F_X \arrow[d, dashed, "\varphi"] & & F_X \arrow[d, dashed, "\exists! \varphi"] \\ & G & & G \end{tikzcd}$$

where $i : X \hookrightarrow F_X$ is the inclusion of X into F_X. Here, the functor is implicitly the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ which simply maps a group to its underlying set $(S, \cdot) \mapsto S$.

3 Back to Algebra

Algebra at its core is about objects, maps, and subobjects. Our goal is to analyze F/I (a free group quotient by relation). For example, $F(a,b)/(a^2 = 1, b^2 = 1, aba = bab) = \mathfrak{S}_3$. Notice that in this relation, $(ab)(ab) = abab = aaba = ba$. Therefore, we can consider the following questions:

- given a word, is it in its most simplified form?
- given two words, are they the same?
- prove when two quotients are the same

Definition 3.1 (Subgroup)**.** A subgroup $H \leq G$ of a group G is a subset $H \subseteq G$ which is closed under the group operation.

Definition 3.2 (Normal Subgroup)**.** A subgroup $N \leq G$ is a normal subgroup (denoted $N \trianglelefteq G$) if it is stable under conjugation. That is, for any $n \in N$ and $g \in G$: $gng^{-1} \in N$.

Definition 3.3 (Group Homomorphism)**.** $f : A \to B$ is a group homomorphism between groups A and B if $f(a \cdot_A b) = f(a) \cdot_B f(b)$ for all $a \in A$ and $b \in B$. The kernel of f is defined to be: $\mathrm{Ker}(f) := \{a \in A | f(a) = 1_B\}$

$\mathrm{Ker}(f)$ is a normal subgroup $\mathrm{Ker}(f) \trianglelefteq A$ because for any $n \in \mathrm{Ker}(f)$ and $a \in A$:

$$f(ana^{-1}) = f(a) \cdot f(n) \cdot f(a^{-1}) = f(a) \cdot 1_B \cdot f(a)^{-1} = 1_B$$

So, $ana^{-1} \in \mathrm{Ker}(f)$.

Now look at $N = \langle a^2, b^2, abab^{-1}a^{-1}b^{-1} \rangle \trianglelefteq F(\{a,b\})$. N is the smallest normal subgroup containing the given elements. The quotient group F/N is defined as follows:

Definition 3.4. Given $N \trianglelefteq G$, define $g \sim g'$ if $g(g')^{-1} \in N$ ($[g]$ is an equivalence class, i.e. the coset gN). Then, the quotient group is $G/N = G/\sim$.

Proposition 3.5. *G/N is indeed a group with the operation $[a] \cdot_{G/N} [b] = [a \cdot_G b]$.*

Proof. First it is necessary to show well-definedness. Let $a \sim a'$ and $b \sim b'$. Then,

$$(ab)(a'b')^{-1} = ab(b')^{-1}(a')^{-1} = \underbrace{ab(b')^{-1}a^{-1}}_{\in N}\underbrace{a(a')^{-1}}_{\in N}$$

Thus, $ab \sim a'b'$. Associativity follows from the definition: $[(ab)c] = [a(bc)]$. The identity is $[1]$ since $[1] \cdot [a] = [1 \cdot a] = [a]$. The inverse of $[a]$ is $[a^{-1}]$ since $[a] \cdot [a^{-1}] = [aa^{-1}] = [1]$. Hence, G/N is a group. $\square$

Theorem 3.6 (First Isomorphism Theorem)**.** *If $f : G \twoheadrightarrow H$ is a surjective homomorphism, then we have $H \cong G/\mathrm{Ker}(f)$, with the isomorphism $\varphi : [g] \mapsto f(g)$.*

Proof. First, to show φ is well-defined, let $g \sim g'$ so $f(g(g')^{-1}) = 1_H$. Thus,

$$f(g) = f(g)f(g')^{-1}f(g') = f(g(g')^{-1})f(g') = 1_H \cdot f(g') = f(g')$$

Assume $\varphi([g]) = \varphi([g'])$ so $f(g) = f(g')$. This means $1_H = f(g)f(g')^{-1} = f(g(g')^{-1})$ so $g \sim g'$. Thus, $[g] = [g']$ meaning φ is injective. For any $h \in H$, consider $[f^{-1}(h)]$. We have: $\varphi([f^{-1}(h)]) = f(f^{-1}(h)) = h$ so φ is surjective. Now, for any $[g], [g'] \in G/\mathrm{Ker}(f)$:

$$\varphi([g])\varphi([g']) = f(g)f(g') = f(gg') = \varphi([gg']) = \varphi([g][g'])$$

Thus, φ is an isomorphism. $\square$

Back to our free group, in this case, consider $F(H) \twoheadrightarrow_f H$ by composing the symbols of the word with composition in H, then $H \cong F(H)/\operatorname{Ker}(f)$.

3.1 Something about other group theory we won't go through

Basically group theory is about those we discussed above, Lagrange theorem, Sylow theory, Jordan Holder, ...etc. We're not gonna spend time on these but somehow we started talking about derived series. The below is a side note without context to what we did in free group.

Given G a group, a commutator is $G^{(1)} = [G : G] = \{xyx^{-1}y^{-1}, x, y \in G\}$, and $G^{(2)} = [G^{(1)} : G^{(1)}]$. Then, the composition series is $\cdots \trianglelefteq G^{(2)} \trianglelefteq G^{(1)} \trianglelefteq G^{(0)}$ with each $G^{(i)}$ being a maximal proper normal subgroup of $G^{(i+1)}$. Jordan Holder states that any two composition series of a given group are equivalent. That is, they have the same composition length and the same composition factors, up to permutation and isomorphism

There are other stuff we'd like to go over (maybe?) in group theory: cyclic groups, $\mathbb{Z}_p$, dihedral group $D_n = F(r,s)/\{r^n = s^2 = (sr)^2 = 1\}$, which is the symmetries on $n-$gon (will see through representation theory)

4 (Linear) Representations of Groups

Consider V, a vector space (finite dimensional for now but can be infinite dimensional), a field $\mathbb{F}$ (usually $\mathbb{C}$ but mathematicians are also interested in $\mathbb{F}_p$, p-adic groups, $\mathbb{R}$), $\mathrm{GL}(V)$, the invertible linear transformation $V \to V$. (can think of $V = \mathbb{F}^n$, since $T \in \operatorname{Hom}(V, W) \hookrightarrow M_{m\times n}$, so $T \in \mathrm{GL}(V) \hookrightarrow M_{n\times n}, \det(M) \neq 0$. Later on, we'd look at $V/\mathbb{F}$ to learn more about representation. Before that, we define what is a linear representation and give examples.

Definition 4.1 (linear representation)**.** A linear representation of a group G is a homomorphism $f : G \to \mathrm{GL}(V)$ for some finite dimensional vector space V over a field $\mathbb{F}$. [Idea: we're linearizing groups]

Example 4.2 (trivial representation)**.** $f : G \to \mathrm{GL}(V)$, where $g \mapsto \mathrm{id}_V$.

Example 4.3 (defining representation)**.** $\mathfrak{S}_n$, the symmetric group on n elements, consists of all permutations of $\{1, \dots, n\}$. Example of composition in $\mathfrak{S}_n$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}.$$

In defining representation, it's written as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 4.4 (sgn representation)**.** $f : \mathbb{G}_n \to \mathbb{C}^\times = \mathrm{GL}(1)$, where $\sigma \mapsto \operatorname{sgn}(\sigma)$.

Next time: examples of representation.
Sidenote:

- trace of determinants is character and they tell a lot.
- Prof. Sergey Fomin (Prof. Yong's advisor) works on totally positive matrix: the determinant of every square submatrix is a positive number.
- we cam talk about determinants coordinate free as well ($\wedge^n V$).