05/26/22 Notes

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- Next meeting time: 10am Friday at AH 245

1 Intersection between Representation and Algebraic geometry

Theorem 1.1. *Given k an algebraically closed field, $f : GL_n(k) \to GL_m(k)$ a rational representation, then there exists a polynomial representation g such that $f(A) = \frac{1}{det(A)} g(A)$.*

Definition 1.2 (affine algebraic variety)**.** For a given field $\mathbb{K}$ and any set of polynomials $S \subseteq \mathbb{K}[x_1, \ldots, x_n]$, define

$$V(S) := \{x \in \mathbb{K}^n | p(x) = 0, \forall p \in S\}.$$

It is standard to prove $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ is the ideal generated by $\langle S \rangle$ in $\mathbb{K}[x_1, \ldots, x_n]$.

Definition 1.3 (ideal generated by points)**.** A dual definition is to look at the ideal generated by points, where the ideal consists of polynomials vanishing on a given set. Define

$$I(V) := \{p \in \mathbb{K}[x_1, \ldots, x_n] | p(x) = 0, \forall x \in V\}.$$

Definition 1.4 (radicals)**.** Define radicals $r(I)$ (or $\sqrt{I}$) of an ideal I in a commutative ring R to be

$$r(I) := \{f \in R | \exists n \in \mathbb{Z}^+, f^n \in I\}$$

Definition 1.5 (Hilbert's Nullstellensatz)**.** Given $\mathbb{K}$ which is an algebraically closed field, and an ideal $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we have

$$I(V(J)) = \sqrt{J}$$

We see that $r(I) \subseteq I(V(I))$ for every ideals $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ easily, by factoring polynomials into linear factors. The other way is less obvious. This powerful theorem gives a complete description of those "secret" equations that vanishes on $V(I)$. For many equations until today, it still remains to be a mystery what are those "secret" equations.

Example 1.6. Consider a $2 * 2$ matrice M = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbb{C}$. What are those equations satisfied by a,b,c, and

d for $M^2 = 0$. A direct computation gives us four equations

$$a^2 + bc = 0$$
$$ab + bd = 0$$
$$ac + cd = 0$$
$$bc + d^2 = 0$$

In this case, it is easy to solve for other equations. But just as an illustration, when a matrix M satisfies $M^2 = 0$, its trace must be 0 (M^2 has 0 eigenvalues, where $\lambda_i^2 = 0$ is an eigenvalue of M^2). A hidden linear equation that can't be generated by the four quadratic equations.

Open Problem 1.7. *What are the polynomial constraints on the entries besides the obvious polynomials for commuting matrices, i.e.* $[A, B] = AB - BA = 0$

2 Representation Theory

Recall some definitions and theorems

- given G a finite group, $f : G \to GL(V)$ over $\mathbb{C}$, the **character** $\chi_f(g) = tr(f(g))$.
- $class(G)$ is a vector space containing χ_f.
- $(\phi|\psi) = \frac{1}{|G|} \sum_{t \in G} \phi(t)\overline{\psi(t)}$.
- $(\chi|\chi') = \begin{cases} 0, & \text{if } \chi \text{ and} \chi' \text{are not isomorphic;} \\ 1, & \text{otherwise.} \end{cases}$
- $(\chi|\chi') = 1 \iff \chi$ is irreducible.
- character determines representations up to isomorphism $\chi_V = \chi'_{V'} \iff V \cong_G V'$.

Follows from the example last time, today we introduce the **regular representation.**

Definition 2.1 (Regular representation)**.** Given $V = \mathbb{C}[G]$ with basis $\{e_g : g \in G\}$, where $te_g = e_{tg}, t \in G$.

Definition 2.2 (Regular representation cited from Serre's)**.** Let g be the order of G, and let V be a vector space of dimension g, with a basis $(e_t)_{t \in G}$ indexed by the elements t of G. For $s \in G$, let p_s be the linear map of V into V which sends e_t to e_{st}; this defines the **regular representation.**

Proposition 2.3 (P5)**.** *The character of the regular representation is given by the formulas:*

$$\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = id \\ 0 & o.w. \end{cases}$$

Proof. The multiplicity of the time Suppose $g \neq 1$, then there is no fixed point under the action of g ($gv'_g = v'_g$ $\leftrightarrow$gg'=g' $\leftrightarrow g = 1$), thus the permutation matrix of g has trace 0. If $g = 1$, the matrix is identity and the trace of it is the dimension of the representation that is the order of the group G. $\square$

Corollary 2.4 (C2)**.** *If* $\mathbb{C}[G] \cong_G \bigoplus_i w_i$*, where* w_i *is irreducible decomposition and* $deg(w_i) = n_i$*. Then,* *(i) The degrees n_i satisfy $\sum_{i=1}^{h} n_i^2 = |G|$.*

(ii) If $s \neq 1$, $\sum_{i=1}^{h} n_i \chi_i(s) = 0$.

Proof. Let r_G denotes the character of regular representation. Since for any irreducible character χ, $(\chi|r_G) = \frac{1}{|G|}\sum_{g\in G}\chi(g)r_G(g^{-1}) = \chi(1)\frac{r_G(1)}{|G|} = \chi(1)$, the multiplicity of an irreducible representation in the decomposition of regular representation is exactly the degree of the irreducible representation. Then, since character behaves nicely to direct sum, $r_G(s) = \sum n_i\chi_i(s)$ for all $s \in G$. To prove (a), take $s = 1$. To prove (b), take $s \neq 1$. □

Now, we want to show that the irreducible class of G, $\chi_1, \dots, \chi_h$ form an orthonormal basis of $class(G)$. Recall that $\chi_1, \dots, \chi_h$ form an orthonormal set (reference: ch.2.3 T3), so we are left to show they generate $class(G)$. First, we prove the following proposition.

Proposition 2.5 (P6)**.** *Let $f \in Class(G)$, $p : G \to GL(V)$ be a linear representation of G. Define $p_f := \sum_{t\in G} f(t)p_t \in End(V)$. If (p, V) is irreducible of degree n and character χ, then p_f is a homothety ($p_f = \lambda Id_v$) of ratio*

$$\lambda = \frac{1}{n}\sum_{t\in G} f(t)\chi(t) = \frac{|G|}{n}(f|\chi^*), \textit{where } \chi^*(g) = \overline{\chi(g)} = \chi(g^{-1}).$$

P6. We would like to use Schur's lemma II for the proof. (Recall the statement: Consider $V = W$ finite-dimensional over an algebraically closed field. Let ρ_V and ρ_W be irreducible representations of G on V and W. Then if $\rho_V = \rho_W$, the only nontrivial G-linear maps are the identity, and scalar multiples of the identity.) To apply Schur's lemma, NTS ρ_f is a G-equiv endomorphism $V \to V$. That is, $\rho_t\rho_s = \rho_s\rho_t, \forall s \in G$. Here,

$$\begin{aligned}\rho_s^{-1}\rho_t\rho_s = \rho_s^{-1}\sum_{t\in G} f(t)\rho_t\rho_s &= \sum_{t\in G} f(t)\rho_{s^{-1}ts}, \text{let } u = s^{-1}ts,\\ &= \sum_{u\in G} f_u\rho_u = \rho_f.\end{aligned}$$

So, we can apply Schur's lemma, given $\rho_f = \lambda Id_\nu$, $tr(\rho_f) = n\lambda$.

We can also obtain the trace with its formula,

$$tr(\rho_f) = tr(\sum_{t\in G} f(t)\rho_t) = \sum_{t\in G} f(t)\chi(t) = |G|(f, \chi^*).$$

Hence, $n\lambda = |G|(f|\chi^*)$. □

With the proposition, we can proceed to prove our goal.

Theorem 2.6 (T6)**.** *The irreducible class of G, $\chi_1, \dots, \chi_h$ form an orthonormal basis of $class(G)$.*

Proof. Given an orthonormal set $\chi_1, \dots, \chi_n$, consider general V, with the Gram-Schmidt process, one can extend such set to an orthonormal basis. Therefore, we only need to show "$\chi_1, \dots, \chi_k$ an orthonormal basis for f if $f \in class(C)$ has $(f|\chi_i^*) = 0, \forall i$, then $f \equiv 0$.

Let $f \in class(G)$, ρ be any linear representation, define $\rho_f = \sum_{t\in G} f(t)\rho(t)$ (as in the prop), then when ρ is irreducible, by the proposition, $\rho_f \equiv 0$. When ρ is reducible, by Maschke's theorem, $\rho_f \equiv 0$.

Lastly, consider ρ as a regular representation, $\rho_f(e_t d) = \sum_{t\in G} f(t)\rho_t e_{td} = \sum_{t\in G} f(t)e_t$, This implies $f(t) = 0, \forall t \in G$. □

Corollary 2.7 (T7)**.** *The number of irreducible representations of G (up to isomorphism) is equal to the number of classes of G.*

3 Relationship to Standard Young Tableaux

For $G = S_n$, $|G| = n!$, $\mathbb{C}[S_n] \cong_G \oplus_{i=1}^{n} n_i w_i \cong \oplus_{\lambda \vdash n_\lambda w_\lambda}$.

By C2(1), $n! = \sum_\lambda n_\lambda^2$. Also, the number of Standard Young Tableaux given shape $\lambda = f^\lambda = n_\lambda$, implying $n! = \sum_\lambda n_\lambda^2$. This is called the Robinson–Schensted correspondence, which is a bijective correspondence between permutations and pairs of standard Young tableaux of the same shape. This correspondence has been generalized in numerous ways, notably by Knuth to what is known as the Robinson–Schensted–Knuth correspondence, and a further generalization to pictures by Zelevinsky [wiki].

Something interesting to think about is how to give a combinatorial proof for the hook-length formula.

Definition 3.1 (Hook Length Formula)**.** The hook length formula is the number of standard Young tableaux of shape λ, denoted by f^λ as

$$f^\lambda = \frac{n!}{\prod h_\lambda(i,j)}.$$

There is a direct bijetive proof given by Novelli, Pak, and Stoyanovskii in 1997. However, something else interesting to consider (a possible alternate proof) is the balanced tableaux.

Definition 3.2 (hook height, hook width, hook length)**.** The *hook height* $h_H(i,j)$ of cell (i,j) is the number of cells below (and including) (i,j) in H_{ij}. Similarly, the *hook width* $h_W(i,j)$ of (i,j) is the number of cells to the right of (and including) (i,j). The *hook length* $h(i,j)$ is defined to be the cardinality of H_{ij}. That is,

$$h(i,j) = h_H(i,j) + h_W(i,j) - 1.$$

Definition 3.3 (hook rank)**.** The hook rank $r(i,j)$ of label t_{ii} is the number of labels $t_{i'j'}$ in H, which are less than or equal to t_{ik}.

Definition 3.4 (balanced tableau)**.** An ordinary tableau is *balanced* of $r(i,j) = h_H(I,J)$ for all cells (i,j).

Balanced tableaux is first defined and interested to Edelman and Greene because of Stanley's conjecture and proof [1982, 1984] on the number of weak Bruhat order of the symmetric group S_n. Though Stanley proved his conjecture, he did not yield an explicit correspondence between maximal chains and standard tableaux. Thus, Edelman and Greene defined the balanced tableuax to encode (inversions) of maximal chain and proved several bijections combinatorially.

Note that we mention this because of the following theorem. Since the number of standard young tableaux is the number of balanced tableaux, one might be able to use the balance tableaux to derive a combinatorial proof for the hook length formula.

Theorem 3.5 (Edelman and Greene, 1987)**.** *Given λ, the number of SYT = the number of balanced tableaux.*

4 Onward to tensor product

Next time, we would discuss tensor product. For now, we would give a sneak peak on this topic.

Given V, W two set, we can find a cartesian product $V \times W$ and a bilinear map $V \times W \to U$. Our aim is to turn bilinear map into linear map. To do so, define a tensor product $V \bigotimes W$ s.t. $\exists!$ linear map h which commutes the diagram.

$$\begin{tikzcd} V \times W \arrow[r, "bilinear"] \arrow[d, "bilinear"] & V \otimes W \arrow[dl, dashed, "\exists! h \text{ linear}"] \\ U & \end{tikzcd}$$

We would introduce formal definition and different properties/actions later. For now, just note that G acts on $V \bigotimes W$ by $g(v \otimes w) = g(v) \otimes g(w)$.

This leads us to discussion of tensor product decomposition. In S_n, the coefficient is Kronecker coefficient, denoted ad $g^{\lambda}_{\mu\nu}$. That is, Kronecker coefficient describe the decomposition of the tensor product of two irreducible representations of a symmetric group into irreducible representations.

$$S_\lambda \otimes S_\mu = \oplus_{\mu \vdash n} S_\mu^{\oplus g(\lambda,\mu,\nu)}, \text{where } g(\lambda,\mu,\nu) \text{ the multiplicity of } S_\nu \text{ in } S_\lambda \otimes S_\mu.$$

Now consider the tensor product of GL_n representation, we obtain the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$, where

$$V_\lambda \otimes V\mu = \oplus_{\lambda \vdash |\mu|+|\nu|} V_\lambda^{\oplus c^{\lambda}_{\mu\nu}}$$

As for our very first discussion today on the theorem, we can rewrite the theorem in terms of tensor product as below

Theorem 4.1. *If V is a rational representation of GL_n, then $V \cong det^{\otimes k} \otimes V'$, where k is the algebraically closed field and V' is a polynomial representation.*