05/27/22 Notes

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- Next meeting time: 10am on Wednesday

Today's topic: finite groups, character theory, poset topology

1 Invariant Theory

Consider S_n acts on $\mathbb{C}[x_1, \dots, x_n]$, $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ is the ring of invariant polynomials.

Example 1.1. symmetric polynomials; non-example: $x_1^2 x_2$.

Theorem 1.2 (Newton)**.** $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ *is generated by elementary symmetric polynomials, where for n general, $k \leq 0$, the symmetric polynomials are*

$$e_k(X_1, \dots, X_n) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} X_{j_1} \cdots X_{j_k}.$$

Note that,

$$\begin{aligned}(x - x_1)(x - x_2) \dots (x - x_n) =& x^n - (x_1 + x_2 + \dots + x_n)x^{n-1} + (x_1x_2 + x_1x_3 + \dots)x^{n-2} \pm \dots \pm x_1x_2 \dots x_n \\ =& x^n - e_1x^{n-1} + e_2x^{n-2} \pm \dots .\end{aligned}$$

Theorem 1.3 (Newton)**.** *Every symmetric polynomial S is a polynomial in the variables $e_1, \dots, e_n$.*

Example 1.4. For $n = 2$, $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = e_1^2 - 2e_2$.

Given G finite group, G acts on $\mathbb{C}[x_1, \dots, x_n]$, the ring of invariants $\mathbb{C}[x_1, \dots, x_n]^G$ is A_n (even permutation, subgroup of S_n where $sgn(w) = 1$.)

1.1 Invariant Theory in 19th century

All these traced back to Paul Albert Gordan (27 April 1837 – 21 December 1912), who was known as "the king of invariant theory". His most famous result is that the ring of invariants of binary forms of fixed degree is finitely generated

Conjecture 1.5. *Given G finite, there exists finitely many generators.*

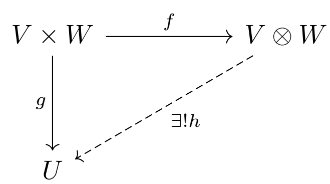
Afterwards, David Hilbert's proof of Hilbert's basis theorem (ideal $\subseteq k[x_1, \dots, x_n]$ is finitely generated) vastly generalized Gordan's result on invariants. This essentially opened the field/discussions of commutative algebra and free resolution. Note that Hilbert's work also works for compact Lie group over $\mathbb{C}$. The recent open problem is, what about characteristic p?

Back to our conjecture, we can consider S_n acting on $\mathbb{C}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle$ is a regular representation (Need to check if well-defined). We can also consider $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$. Let $S_{n\sigma}$ be diagonalization such that $S_{n\sigma}\rho(x_1, dots, x_n, y_1, \dots, y_n) = \rho(x_{\sigma(1)}, dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)})$.

The Garsia-Haiman theory says when $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}$ is finite and of invaraint known, then S_n acts on $\mathbb{C}[x; y]/\langle \text{invariants} \rangle$ is the **Macdonald polynomials**.

2 Tensor Product

Let V and W be arbitrary vector spaces. We wish to define the *tensor product* $V \otimes W$. As motivation, we want to be able to convert a bilinear map $V \times W \to U$ into a linear map $V \otimes W \to U$. Specifically, $V \otimes W$ should satisfy the following universal property: There is a bilinear map $f : V \times W \to V \otimes W$ such that for any vector space U and any bilinear map $g : V \times W \to U$ there exists a unique linear map $h : V \otimes W \to U$ such that the following diagram commutes:



This defines the tensor product up to a unique isomorphism, but it still needs to be shown that a vector space satisfying the universal property exists at all.

Definition 2.1 (Free Vector Space)**.** The free vector space $F(X)$ of a set X over a field $\mathbb{K}$ is the set of all formal finite linear combinations of elements of X:

$$F(X) = \left\{ \sum_{i=1}^{n} a_i x_i \,\middle|\, a_i \in \mathbb{K},\, x_i \in X,\, n \in \mathbb{Z}^+ \right\}$$

To make $F(X)$ into a vector space, addition and multiplication are defined pointwise, i.e.

$$(a_1 x_1 + \cdots + a_n x_n) + (b_1 x_1 + \cdots + b_n x_n) = (a_1 + b_1)x_1 + \cdots + (a_n + b_n)x_n$$
$$\alpha \cdot (a_1 x_1 + \cdots + a_n x_n) = (\alpha a_1)x_1 + \cdots + (\alpha a_n)x_n$$

Now, consider the free vector space $F(V \times W)$. From this, we can construct the tensor product by quotienting-out a subspace. Define the subspace $R \subseteq F(V \times W)$ as follows:

$$R = \operatorname{span}\left\{ \begin{array}{l} (v + v', w) - (v, w) - (v', w), \\ (v, w + w') - (v, w) - (v, w'), \\ a(v, w) - (av, w), \\ a(v, w) - (v, aw) \end{array} \,\middle|\, v, v' \in V,\ w, w' \in W,\ a \in \mathbb{K} \right\}$$

Definition 2.2 (Tensor Product)**.** The tensor product of V and W is defined to be: $V \otimes W = F(V \times W)/R$.

An element $[(v, w)] \in V \otimes W$ is typically denoted instead by $v \otimes w$. The vectors included in R give the operation $\otimes$ the following properties:

- $(v + v') \otimes w = v \otimes w + v' \otimes w$
- $v \otimes (w + w') = v \otimes w + v \otimes w'$
- $a(v \otimes w) = (av) \otimes w$
- $a(v \otimes w) = v \otimes (aw)$

The function f in the universal property is then given by $f : (v, w) \mapsto v \otimes w$.

Proposition 2.3. *$V \otimes W$ satisfies the universal property of the tensor product.*

Proof. That f is bilinear follows directly from the above properties of $\otimes$. Given a bilinear map $g : V \times W \to U$, define the linear map $h : V \otimes W \to U$ by $h(v_i \otimes w_j) = g(v_i, w_j)$ extended linearly where $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are the respective bases of V and W. We have:

$$h(f(v_i, w_j)) = h(v_i \otimes w_j) = g(v_i, w_j)$$

And since $h \circ f$ and g are bilinear, $g = h \circ f$. Now, suppose two such functions $h_1 \neq h_2$ exist satisfying $g = h_1 \circ f = h_2 \circ f$. We have:

$$h_1(v_i \otimes w_j) = h_1(f(v_i, w_j)) = h_2(f(v_i, w_j)) = h_2(v_i \otimes w_j)$$

And since h_1 and h_2 are linear, $h_1 = h_2$. Therefore, the linear map h is unique. □

The tensor product of two R-modules can be defined completely analogously and is denoted $V \otimes_R W$. Technically, the base field of the vector spaces in a regular tensor product should also be specified, i.e. $V \otimes_{\mathbb{F}} W$.

Proposition 2.4. *For finite dimensional vector spaces V and W, we have:* $\mathrm{Hom}(V, W) \cong V^* \otimes W$.

Proof. Given a linear transformation $f \in \mathrm{Hom}(V, W)$, it can be expressed in the basis of W as:

$$f(v) = a_1(v) \cdot w_1 + a_2(v) \cdot w_2 + \cdots + a_m(v) \cdot w_m$$

Since f is linear, each a_i must be a linear functional. Define a function $T : \mathrm{Hom}(V, W) \to V^* \otimes W$ by:

$$T(f) = a_1 \otimes w_1 + a_2 \otimes w_2 + \cdots a_m \otimes w_m$$

T is a linear map because $(a_i + cb_i) \otimes w_i = (a_i \otimes w_i) + c(b_i \otimes w_i)$. T is surjective because each $v_i^* \otimes w_j$ is mapped to by $f(x) = v_i^*(x) \cdot w_j$. Finally, if $T(\sum_{i=1}^m a_i w_i) = T(\sum_{i=1}^m b_i w_i)$ then $\sum_{i=1}^m a_i \otimes w_i = \sum_{i=1}^m b_i \otimes w_i$ so $a_i = b_i$ for each i. Hence, $\sum_{i=1}^m a_i w_i = \sum_{i=1}^m b_i w_i$. Hence, T is injective. Therefore, $\mathrm{Hom}(V, W) \cong V^* \otimes W$. □

Proposition 2.5. *If V and W are G-modules, then $V \otimes W$ is a G-module under the action $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$ extended linearly.*

Proof. For any $g, g' \in G$, we have:

$$(gg') \cdot (v \otimes w) = ((gg') \cdot v) \otimes ((gg') \cdot w) = (g \cdot g' \cdot v) \otimes (g \cdot g' \cdot w) = g \cdot g' \cdot (v \otimes w)$$

Therefore, $V \otimes W$ is a G-module. □

If $f^1 : G \to \mathrm{GL}(V)$ and $f^2 : G \to \mathrm{GL}(W)$ are the corresponding representations given in matrix form by:

$$f^1(g) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad f^2(g) = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{bmatrix}$$

Then the representation $f^1 \otimes f^2$ corresponding to $V \otimes W$ will be given by the Kronecker product:

$$(f^1 \otimes f^2)(g) = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1m} & & a_{1n}b_{11} & \cdots & a_{1n}b_{1m} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{11}b_{m1} & \cdots & a_{11}b_{mm} & & a_{1n}b_{m1} & \cdots & a_{1n}b_{mm} \\ & \vdots & & \ddots & & \vdots & \\ a_{n1}b_{11} & \cdots & a_{n1}b_{1m} & & a_{nn}b_{11} & \cdots & a_{nn}b_{1m} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{n1}b_{m1} & \cdots & a_{n1}b_{mm} & & a_{nn}b_{m1} & \cdots & a_{nn}b_{mm} \end{bmatrix}$$

Here, the matrix is given with the basis of $V \otimes W$ in the order $\{v_1 \otimes w_1, \ldots, v_1 \otimes w_m, \cdots, v_n \otimes w_1, \ldots, v_n \otimes w_m\}$ where $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are the respective bases of V and W. This can be verified by simply expanding $(g \cdot v_i) \otimes (g \cdot w_j)$ using the properties of $\otimes$.

Proposition 2.6. *Given two representations f^1 and f^2 of the same group, we have: $\chi_{f^1 \otimes f^2} = \chi_{f^1} \cdot \chi_{f^2}$.*

Proof. Using the matrix given above:

$$\chi_{f^1 \otimes f^2}(g) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ii} b_{jj} = \left(\sum_{i=1}^{n} a_{ii} \right) \cdot \left(\sum_{j=1}^{m} b_{jj} \right) = \chi_{f^1}(g) \cdot \chi_{f^2}(g)$$

because the indices i and j are independent of each other. □

This proposition allows us to relate the Kronecker Problem to character tables. Because characters respect tensor products and direct sums:

$$W_\lambda \otimes W_\mu = \bigoplus_\nu k^\nu_{\lambda\mu} W_\nu \qquad \rightsquigarrow \qquad \chi_\lambda \cdot \chi_\mu = \sum_\nu k^\nu_{\lambda\mu} \chi_\nu$$