A=B: The Case for CAS and Research in High School Mathematics

by Justin Burner

Having excelled in my high school mathematics classes, I was more than willing to declare myself a math major upon entering college. I soon discovered that I had been completely misled by my former teachers. Secondary math was not *real* math; proofs from geometry class had not been *real* proofs. Paul Lockhart, in his infamous "Mathematician's Lament" essay, describes exactly why many students find themselves in this exact predicament: "Many a graduate student has come to grief when they discover, after a decade of being told they were 'good at math,' that in fact they have no real mathematical talent and are just very good at following directions. Math is not about following directions, its about making new directions" [Lockhart 2008, p. 6]. I could certainly factor polynomials, manipulate trigonometric identities, and calculate integrals with ease – all by following basic algorithms – without truly understanding how or why these mathematical concepts worked, or even the historical significance of the theories behind these algorithms. Furthermore, contrary to the work done by undergraduate and graduate math students, most high school students do little to no individual or collaborative research or investigation.

It is this disconnect between secondary school math and "real" math (particularly in the United States math education track of algebra-geometry-trigonometry-calculus) that leaves many high school students bored and college students unprepared. Technologies like slide rules, scientific calculators, and graphing calculators present opportunities for students to dig deeper into mathematical concepts and solve problems which in the past seemed too difficult or inaccessible. The implementation of these devices, however, has been overall lackluster.

These devices are primarily used by students to facilitate calculations and graphing, which leads many students to develop a dependence on technology while not necessarily learning anything different than before. Within the past 10 years, the development and increased accessibility of computer algebra systems (CAS) like Mathematica (and its companion site WolframAlpha) allows students of algebra and calculus to complete nearly any exercise in symbolic manipulation by entering the problem into a computer and retrieving an answer – sometimes including intermediate steps and explanations. "The worst may happen if the student embarks upon computations or constructions without having *understood* the problem" [Pólya 1945, p. 5]. George Pólya's worst fears may come true if secondary mathematics teachers do not recognize the educational power of CAS and the necessity to change not only the mathematical content being taught in high schools but also the methods through which students experience this content.

Different computer algebra programs can be found on a variety of devices that many students can already access in the classroom - particularly with the increase in one-to-one classroom initiatives that pair each student with a laptop or tablet. CASs are available on graphing calculators (TI-Nspire), desktop programs (Geogebra, Mathematica), and web applications (Desmos, WolframAlpha). Aside from solving problems that require symbolic manipulation, a CAS enables students to quickly investigate patterns that arise in more abstract problems involving binomial expansion and families of functions. Stephen Wolfram himself describes how a CAS can be used to develop students' sense for abstract reasoning and problem solving: "'A lot of it boils down to, Can you use these tools to get intuition rather than just mechanical skills?' said Mr. Wolfram. 'After kids see Mathematica, or now WolframAlpha, some fraction of them become curious and wonder, How does that actually do that?' " [Young 2009]. A basic understanding of computer languages and the syntax of CAS, paired with a teacher willing to develop investigatory activities, can greatly increase students' ability to conceptualize abstract algebraic or trigonometric rules. This atmosphere of discussion, research, and investigation – while present in high school science, social studies, and English classes – needs to become an integral part of our secondary mathematics curriculum.

The idea of using a CAS in the classroom, as well as the use of simpler technologies like basic calculators, is not without its opponents. A common cry heard from math teachers at all levels is that, due to the omnipresence of calculators, students do not know their basic math facts. Others worry that computer algebra programs allow students to lose the mastery of symbolic manipulation inherent in our current algebra classes. "A current colleague once told me that his biggest concern about using CAS is that it reduces many skills into one skill: All limit problems become the same when entered into CAS. I would not say, however, that students who use CAS to solve a limit problem have no idea what they are doing, just as I would not say that students who correctly perform all the algebraic steps required to rationalize the numerator or denominator of a rational function in order to find its limit have a true and deep understanding of what they are doing" [Freda 2008]. Once again, the ability to follow directions does not necessarily translate into real mathematical insight.

Teachers who believe that algebra is solely about symbolic manipulation are on one hand misleading their students and on the other hand perpetuating a myth to the public that algebra is not relevant. Citing mathematics as a prominent reason for many high school and college failure stories, along with seemingly unnecessary math requirements for several technical degree programs, Andrew Hacker, professor emeritus of political science at Queens

College in New York, questioned whether or not we should be teaching algebra at all in secondary schools. "Being able to detect and identify ideology at work behind the numbers is of obvious use. Ours is fast becoming a statistical age, which raises the bar for informed citizenship. What is needed is not textbook formulas but greater understanding of where various numbers come from, and what they actually convey" [Hacker 2012]. While not in favor of the current college algebra courses being offered, Hacker's article suggests that a change towards more research-based mathematics might be a step in the right direction. A rebuttal from Stanford doctoral student and former secondary math teacher Dan Meyer raises a slightly different question. "Our world is increasingly automated and programmed and if you want any kind of active participation in that world, you're going to need to understand variable representation and manipulation. That's Algebra. Without it, you'll still be able to clothe and feed yourself, but that's a pretty low bar for an education. The more interesting question is, 'How should we define Algebra in 2012 and how should we teach it?' Those questions don't even seem to be on Hacker's radar" [Meyer 2012-07]. The proper implementation of technologies like CASs and graphing calculators can help make algebra relevant as well as provide a platform for mathematical research and discussion.

While computer algebra programs are slowly being introduced to the world of secondary education, they have become an invaluable tool to research mathematicians and post-secondary math students. With the right code and a powerful enough computing environment, we are now able to solve problems which have been historically difficult in terms of both calculation and conceptualization. The details of the computer methods used to solve certain combinatorial and hypergeometric identities are outlined in the book $A{=}B$ by Marko Petkovšek, Herbert Wilf, and Doron Zeilberger, first published in 1996. While focused primarily on proving identities via the implementation of algorithms developed by Sister Mary Celine Fasenmyer and Bill Gosper, *A=B* features many parts which are quite accessible to those not immediately familiar with recurrence relations or hypergeometric sums. The book makes a strong case for the value of computers in mathematics research and tactfully addresses concerns that might arise when discussing computer-assisted proofs.

Up front, the authors of *A=B* stress the importance of understanding the purity of the mathematics and appreciating the art and beauty of hand-written proofs. "The main purpose of this book is to explain how the discoveries and the proofs of hypergeometric identities have been very largely automated. The book is not primarily about computing; it is the mathematics that underlies the computing that will be the main focus. Automating the discovery and proof of identities is not something that is immediately obvious as soon as you have a large computer" [Petkovšek 1997, p. 22]. Computer-assisted problem solving has been met with some resistance, a famous example being the 1976 proof of the four color theorem. *A=B* attempts to assuage any doubts readers may have regarding the inelegance of computer-assisted proofs or the potential inability to check such proofs by hand. "People get unhappy when a computer blinks its lights for a while and then announces a result, if people cannot easily check the truth of the result for themselves. In this book you will be pleased to note that although the computers will have to blink their lights for quite a long time, when they are finished they will give to us people a short certificate from which it will be easy to check the truth of what they are claiming" [Petkovšek 1997, p. 17]. The way that *A=B* presents the history and theory behind these algorithms can help change the minds of those who hold an outdated or inaccurate view of modern mathematics.

This change in the public's conception of mathematics is necessary if we continue to argue in favor of mandatory mathematics education in secondary schools. Pólya's *How To Solve It* makes an effort to catalyze this change through an increase knowledge of research methods and problem solving skills. "Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science. Both aspects are as old as the science of mathematics itself. But the second aspect is new in one respect; mathematics 'in statu nascendi,' in the process of being invented, has never before been presented in quite this manner to the student, or to the teacher himself, or to the general public" [Pólya 1945, p. vii]. The use of CAS-enabled calculators and computer programs can help greatly accelerate this change. "CASs enable users to experiment with algebra, pre-calculus, and calculus, while relying, with caution, on the algebraic accuracy of the system and the correct use of it ... CASs allow users to concentrate on setting up problems, rather than on the details of solving each of them" [Mahoney 2002]. Dan Meyer has been a vocal proponent of making students formulate problems themselves – it allows mathematics to spur conversation and helps students see interdisciplinary connections while improving their argumentation skills.

Being able to implement technologies like CASs in the classroom can be difficult for teachers who are used presenting formulas and assigning exercises. "Students must realize that the CAS produces not just a simulation of the symbolic manipulation that they may have learned to do by hand by also that it offers an opportunity to access different symbolic reasoning paths. The ability to successfully maneuver these paths, however, may not develop naturally. As teachers, we need to find ways to help students develop their abilities to reason symbolically using the power of CAS" [Heid 2002]. Even for students and teachers who are accustomed to using technologies like graphing calculators on a daily basis, transitioning to lessons involving CAS comes with somewhat steep learning curve. One must be familiar with not only the language and commands necessary to use the CAS but also the not-quite-universal method of using computers to type mathematics. Dan Meyer, while discussing failure rates of online math instruction, cites computers as not being a "natural medium" for mathematics. "There's nothing intuitive about pressing Shift + 6 to write an exponent, no inherent connection between the idea and the action ... I'd wager 90% of people reading this already know how to type an exponent on a computer. They believe it's easy enough to teach and I don't think they're wrong. But this is only one instance of a problem with a lot of reach. Notation makes math difficult on a computer. But notation also makes math more powerful and interesting. That tension will be very difficult to resolve and, so far, online math providers have generally resolved it in favor of the computer at the expense of math's interest and power" [Meyer 2013].

By harnessing the mathematical power offered by CAS, we can increase the pace at which research is done and broaden the scope of the types of mathematics which can be taught in secondary and post-secondary classrooms. Donald Knuth, in the forward of A=B, also mentions that computer-based methods can help make mathematics more accessible to a wider audience while eliminating the need to struggle with a difficult combinatorial identity when a "simple" answer may not even exist. "No longer do we need to get a brilliant insight in order to evaluate sums of binomial coefficients, and many similar formulas that arise frequently in practice; we can now follow a mechanical procedure and discover the answers quite systematically ... Notice that the algorithm doesn't just verify a conjectured identity 'A = B.' It also answers the question 'What is A?', when we haven't been able to formulate a decent conjecture" [Petkovšek 1997, p. vii]. Even in high school mathematics, instead of presenting a formula up front and asking students to apply it, a CAS-based lesson can allow students to engage in research, observe patterns, and discover on their own what formulas apply and why they work. The authors of $A{=}B$, however, still include a caveat: "Obviously we cannot claim that the computerized methods are the best for every situation. Sometimes the certificates that they produce are longer and less user-friendly than those that humans might find, for example. But the emergence of these methods has put an important family of tools in the hands of discrete mathematicians, and many results that are accessible in no other way have been found and proved by computer methods" [Petkovšek 1997, p. 50].

This inelegance of computer-generated answers is part of the argument against incorporating these technologies into secondary mathematics classrooms. Not only might students start losing the "core skills" (arithmetic, memorization, and symbolic manipulation) that were substantial elements of the mathematics education of the previous generation, but they might miss out on the art and purity of doing math by hand. $A{=}B$ discusses this early on, using a standard combinatorial proof of the basic identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$. "We must pause to remark that that one is a really nice proof. So as we go through this book whose main theme is that computers can prove all of these identities, please note that we will never claim that computerized proofs are better than human ones, in any sense. When an elegant proof exists, as in the above example, the computer will be hard put to top it. On the other hand, the contest will be close even here, because the computerized proof that's coming up is rather elegant too, in a different way" [Petkovšek 1997, p. 24]. However, as a potential nod to how truly amazing these algorithms can be for a struggling discrete mathematician, the word *never* is accompanied by a footnote stating, "Well, hardly ever." The authors also give a proof of the same identity using generating functions, which were referenced by

George Pólya in the inaugural issue of the Journal of Combinatorial Theory: "I am afraid that it is impossible to give a satisfactory definition of 'Combinatorial Theory' and to trace out a sharp boundary of its subject matter, but mathematicians readily recognize what is of 'combinatorial character.' To delimit the methods for solving combinatorial problems seems not only impossible, but may be also undesirable. Yet we have to mention here a favorite tool of combinatorial mathematics: the generating functions" [Pólya 1966 p. 1]. Though A=B does not complete the *impossible* as described by Pólya, it certainly makes a significant dent while avoiding the *undesirable*. Even in the conclusion of A=B, the authors emphasize the viewpoint that computer-aided mathematics is a stepping stone rather than a destination. "The proofs and the theory here (as elsewhere) are far more important than the identities themselves . . . Let's hope that in the future, computers will supply us humans with many more beautiful identities, that will turn out to be tips of many beautiful icebergs to come. So the moral is that we need both tips and icebergs, since tips by themselves are rather boring (but not the activity of looking for them!), and icebergs are nice, but we would never find them without their tips" [Petkovšek 1997, p. 195]. Indeed, the agreement with Pólya's viewpoint is evident, as the existence of algorithms that could solve all combinatorial problems truly would be *undesirable*, eliminating the need for research and experimentation.

Advances in technology can certainly change the nature of mathematical research and education, and math teachers must be open to this change if we plan on convincing the public that math education is still relevant. "Science advances whenever an Art becomes a Science. And the state of the Art advances too, because people always leap into new territory once they have understood more about the old" [Petkovšek 1997, p. vii]. Slide rules applied the properties of logarithms to greatly increase the types of things which could be calculated without referring to data tables. Scientific calculators proved to be a more accurate, speedy, accessible, and user-friendly tool, particularly when approximating roots and trigonometric values. Graphing calculators allowed students to quickly observe the behavior of multiple functions without the drudgery of making tables and sketching by hand. Now that computer algebra systems are becoming more readily available and inexpensive, secondary mathematics teachers have the opportunity to transform the traditional teachings of symbolic manipulation into interactive, engaging discussions and cross-curricular, research-based projects. "Mathematics is the art of explanation. If you deny students the opportunity to engage in this activity - to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs - you deny them mathematics itself" [Lockhart 2008, p. 5]. The small investment in learning the basics of a CAS and designing lessons that incorporate elements of research and discussion can result in a generation of students who are not averse to problem solving and experimentation.