REDUCED WORD ENUMERATION, COMPLEXITY, AND RANDOMIZATION

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ABSTRACT. A *reduced word* of a permutation w is a minimal length expression of w as a product of simple transpositions. We examine the computational complexity, formulas and (randomized) algorithms for their enumeration. In particular, we prove that the *Edelman-Greene statistic*, defined by S. Billey-B. Pawlowski, has exponentially growing expectation. This is established by a formal run-time analysis of A. Lascoux-M.-P. Schützenberger's *transition algorithm*. The more general problem of Hecke word enumeration, and its closely related question of counting *set-valued standard Young tableaux*, is also investigated. The latter enumeration problem is further motivated by work on *Brill-Noether varieties* due to M. Chan-N. Pflueger and D. Anderson-L. Chen-N. Tarasca.

1. INTRODUCTION

1.1. **Reduced word combinatorics.** Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. Each $w \in S_n$ can be expressed as a product of $\ell(w)$ simple transpositions $s_i = (i, i+1)$, where $\ell(w)$ is the number of *inversions* of w, i.e., pairs $i < j$ such that $w(i) > w(j)$. Such an expression $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ is a *reduced word* for w.

Let $\mathsf{Red}(w)$ be the set of reduced words for w. R. P. Stanley [34] defined a symmetric function F_w such that

$$\#\mathsf{Red}(w) = \text{the coefficient of } x_1 x_2 \cdots x_{\ell(w)} \text{ in } F_w. \tag{1}$$

In connection to *ibid.*, P. Edelman-C. Greene [13, Section 8] proved that

$$\#\mathsf{Red}(w) = \sum_{\lambda} a_{w,\lambda} f^{\lambda}, \quad \text{where} \tag{2}$$

- f^{λ} is the number of *standard Young tableaux* of shape λ, that is, row and column increasing bijective fillings of the Young diagram of λ using $1, 2, \ldots, |\lambda|$. The *hook-length formula* of J. S. Frame-G. de B. Robinson-R. M. Thrall [16] states

$$f^{\lambda} = \frac{|\lambda|!}{\prod_b h_b}, \tag{3}$$

  where the product is over all boxes $b \in \lambda$ and h_b is the *hooklength* of b, i.e., the number of boxes weakly right and strictly below b.
- $a_{w,\lambda}$ counts *EG tableaux*: row and column increasing fillings T of λ such that reading the entries $(i_1, i_2, \ldots, i_{|\lambda|})$ of T along columns, top to bottom, and right to left, gives a reduced word $s_{i_1} \cdots s_{i_{|\lambda|}}$ for w (cf. [10]).

Let $w_0 = n\ n-1\ n-2\ \ldots\ 3\ 2\ 1$ be the unique longest length permutation of S_n (hence $\ell(w_0) = \binom{n}{2}$). R. P. Stanley [34] proved that, in this case, (2) is short:

$$\#\mathsf{Red}(w_0) = f^{(n-1,n-2,\ldots,3,2,1)}; \tag{4}$$

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hence $\#\mathsf{Red}(w_0)$ is computed by (3).

One measure of the brevity of (2) is the *Edelman-Greene statistic* on S_n,

$$\mathsf{EG}(w) = \sum_{\lambda} a_{w,\lambda};$$

this was introduced by S. Billey-B. Pawlowski [5]. From (4), one sees $\mathsf{EG}(w_0) = 1$. Permutations w such that $\mathsf{EG}(w) = 1$ are *vexillary*. These permutations are characterized by 2143-*pattern avoidance*: there are no indices $i_1 < i_2 < i_3 < i_4$ such that $w(i_1), w(i_2), w(i_3), w(i_4)$ are in the same relative order as 2143. For instance, $w = \underline{5}\underline{4}27\underline{8}31\underline{6}$ is not vexillary; the underlined positions give a 2143 pattern. Each such w has *shape* $\lambda(w)$ (defined in Section 2.2). Extending (4), whenever w is vexillary,

$$\#\mathsf{Red}(w) = f^{\lambda(w)}; \tag{5}$$

see, e.g., [29, Corollary 2.8.2]. We prove (Theorem 3.2) that EG is typically large, a consequence of which is:

Theorem 1.1 (Average exponential growth). *$\mathbb{E}[\mathsf{EG}] = \Omega(c^n)$, for some fixed constant $c > 1$.*

1.2. **Computational complexity and transition.** Our proof of Theorem 1.1 applies the *transition algorithm* of A. Lascoux-M. P. Schützenberger [26] (cf. [29, Sections 2.7, 2.8]). This algorithm constructs a tree $\mathcal{T}(w)$ whose root is w and the leaves $\mathcal{L}(w)$ are labelled with vexillary permutations (with multiplicity). With this,

$$\#\mathsf{Red}(w) = \sum_{v \in \mathcal{L}(w)} f^{\lambda(v)}; \tag{6}$$

see Section 2 for details. Different v may give the same $\lambda(v)$. After combining such terms, (6) is the same as (2); see Lemma 3.1.

The (practical) efficiency of (extensions/variations of) transition has been mentioned a number of times. S. Billey [3] calls transition "one of the most efficient methods" to compute Schubert polynomials. See also A. Buch [9, Section 3.4] and Z. Hamaker-E. Marburg-B. Pawlowski [19]. On the other hand, concerning the application of transition to computing the Littlewood-Richardson coefficients [26], A. Garsia [18, p. 52] writes:

> "Curiously, their algorithm (in spite of their claims to the contrary) is hopelessly inefficient as compared with well known methods."

He also refers to transition as "efficient" for a different purpose in his study of $\mathsf{Red}(w)$.

Theorem 1.1 is actually a reformulation of the following result which, as far as we can tell, is the first *formal* complexity analysis of transition:

Theorem 1.2. *$\mathbb{E}(\#\mathcal{L}) = \Omega(c^n)$ for a fixed constant $c > 1$. That is the average running time of transition, as an algorithm to compute $\#\mathsf{Red}(w)$, is at least exponential in n.*

To prove Theorem 1.2 we use that the expected number of occurences of a fixed pattern $\pi \in S_k$ in $w \in S_n$ is $\binom{n}{k}/k!$. Thus for $u = 2143$, this expectation is $O(n^4)$. One shows each step of transition reduces the number of 2143 patterns by $O(n^3)$. Using the graphical description of transition by A. Knutson and the third author [25], a node u of $\mathcal{T}(w)$ has exactly one child u' only if u' has weakly more 2143 patterns than u does. Consequently, $\mathcal{T}(w)$ has $\Omega(n)$ branch points along any root-to-leaf path and thus exponentially many leaves. (In fact, the $c > 1$ from our argument is close to 1.)

Of course, the exponential average run-time of transition does not imply computing $\#\mathsf{Red}(w)$ is hard. Suppose one encodes a permutation w by its *Lehmer code* $\mathsf{code}(w) = (c_1, c_2, \ldots, c_L)$. *What is the worst case complexity of computing* $\#\mathsf{Red}(w)$ *given input* $\mathsf{code}(w)$*?*

L. Valiant [36] introduced the complexity class $\#\mathsf{P}$ of problems that count the number of accepting paths of a non-deterministic Turing machine running in polynomial time in the length of the input. Let FP be the class of function problems solvable in polynomial time on a deterministic Turing machine. It is basic theory that $\mathsf{FP} \subseteq \#\mathsf{P}$.

Observation 1.3. $\#\mathsf{Red}(w) \notin \#\mathsf{P}$. *In particular,* $\#\mathsf{Red}(w) \notin \mathsf{FP}$.

Proof. Let θ_n be the vexillary permutation with $\mathsf{code}(\theta_n) = (n, n)$. Then, using (5),

$$\#\mathsf{Red}(\theta_n) = f^{(n,n)} = C_n := \frac{1}{n+1}\binom{2n}{n}. \tag{7}$$

The middle equality is textbook: there is a bijection between standard Young tableaux of shape (n, n) and Dyck paths from $(0, 0)$ to $(2n, 0)$; both are enumerated by the *Catalan number* C_n. Now, $\#\mathsf{Red}(\theta_n)$ is *doubly* exponential in the input length $O(\log n)$. No such problem can be in $\#\mathsf{P}$ [30, Section 3]. ($\#\mathsf{Red}(w) \notin \mathsf{FP}$ is true from this argument for the simple reason that it takes exponential time just to write down the output.) □

A counting problem $\mathcal{P}$ is $\#\mathsf{P}$-hard if any problem in $\#\mathsf{P}$ has a polynomial-time counting reduction to $\mathcal{P}$. *Is* $\#\mathsf{Red}(w) \in \#\mathsf{P}$-hard*?*

Observation 1.3 is dependent on the choice of encoding. For example, if one encodes a permutation $w \in S_n$ in the inefficient one-line notation, the input takes $O(n \log n)$ space. Since $\ell(w) \leq \binom{n}{2}$ is polynomial in the input length, it follows that $\#\mathsf{Red}(w) \in \#\mathsf{P}$; see [33].

Problem 1.4. *Does there exist an* $n^{O(1)}$*-algorithm to compute* $\#\mathsf{Red}(w)$*?*

It is easy to see that $\#\mathsf{Red}(u) \leq \#\mathsf{Red}(us_i)$ whenever $\ell(us_i) = \ell(u) + 1$. Hence, $\#\mathsf{Red}(w)$ is maximized at $w = w_0$. So, by (4), $\log(\#\mathsf{Red}(w)) \in n^{O(1)}$. Thus, unlike Observation 1.3, there is no easy negative solution to Problem 1.4 (and any negative solution implies $\mathsf{FP} \neq \#\mathsf{P}$, which is a famous open problem). Indeed, in the vexillary case (5), the hook-length formula (3) gives a $n^{O(1)}$-algorithm for $\#\mathsf{Red}(w)$.

1.3. **Hecke words.** Section 4 studies the more general problem of counting $\mathsf{Hecke}(w, N)$, the set of *Hecke words* of length N whose *Demazure product* is a given $w \in S_n$. Here, the role of Stanley's symmetric polynomial is played by the *stable Grothendieck polynomial* defined by S. Fomin and A. N. Kirillov [15]. Using work of S. Fomin and C. Greene [14] and of A. Buch, A. Kresch, M. Shimozono, H. Tamvakis and the third author [10], one has two analogues of the results of Edelman-Greene [13]. However, useful enumeration *formulas* for Hecke words, even when w is vexillary, is a challenge.

As explained by Proposition 4.3, enumerating Hecke words is closely related to the problem of counting $f^{\lambda,N}$, the number of *set-valued tableaux* [8] that are N*-standard* of shape λ. These are fillings T of the boxes of λ by $1, 2, \ldots, N$, where each entry appears exactly once, and if one chooses precisely one entry from each box of T, one obtains a semistandard tableau. For example, if $N = 8$ and $\lambda = (3, 2)$, one tableau is $\begin{array}{|c|c|c|}\hline 1,2 & 4,5 & 8 \\ \hline 3 & 6,7 \\ \cline{1-2}\end{array}$.

By Observation 1.3's reasoning, (7) shows there is no algorithm to compute $f^{\lambda,N}$ that is polynomial-time in the bit-length of the input (λ, N).

Problem 1.5. *Does there exist an algorithm to compute $f^{\lambda,N}$ that is polynomial in $|\lambda|$ and N?*

Clearly, (3) gives a solution when $N = |\lambda|$. Using a theorem of C. Lenart [27], one shows there exists an $|\lambda|^{O(1)}$ algorithm for any λ and where $N = |\lambda| + k$, but only if k is fixed in advance (Proposition 4.5).

Recent work of M. Chan-N. Pflueger [11] and D. Anderson-L. Chen-N. Tarasca [2] motivates study of $f^{\lambda,N}$ in terms of *Brill-Noether varieties*. We remark on two manifestly nonnegative formulas for the Euler characteristics of these varieties (Corollary 4.10).

1.4. **Randomization.** Section 5 gives three randomized algorithms to estimate $\#\mathsf{Red}(w)$ and/or $\#\mathsf{Hecke}(w, N)$ using *importance sampling*. That is, let S be a finite set. Assign $s \in S$ probability p_s. Let Z be a random variable on S with $Z(s) = 1/p_s$. Then $\mathbb{E}(\mathsf{Z}) = \sum_{s\in S} p_s \times \frac{1}{p_s} = \#S$.[1] Using this, one can devise simple Monte Carlo algorithms to estimate $\#S$. The idea goes back to at least a 1951 article of H. Kahn-T. E. Harris [21], who furthermore credit J. von Neumann. The application to combinatorial enumeration was popularized through D. Knuth's article [22] which applies it to estimating the number of self-avoiding walks in a grid. An application to approximating the *permanent* was given by L. E. Rasmussen [31]. More recently, J. Blitzstein-P. Diaconis [6] develop an importance sampling algorithm to estimate the number of graphs with a given degree sequence. We are suggesting another avenue of applicability, to core objects of algebraic combinatorics.

2. The Graphical Transition algorithm

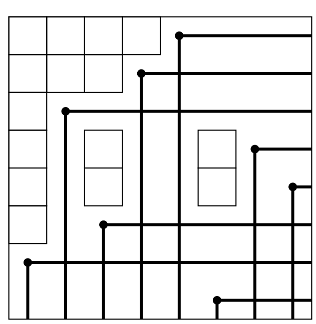
2.1. **Preliminaries.** The *graph* $G(w)$ of a permutation $w \in S_n$ is the $n \times n$ grid, with a $\bullet$ placed in position $(i, w(i))$ (in matrix coordinates). The *Rothe diagram* of w is given by

$$D(w) = \{(i, j) : 1 \leq i, j \leq n,\ j < w(i),\ i < w^{-1}(j)\}.$$

Pictorially, this is described by striking out boxes below and to the right of each $\bullet$ in $G(w)$. $D(w)$ consists of the remaining boxes. If it exists, the connected component involving $(1, 1)$ is the *dominant component*. The *essential set* of w consists of the maximally southeast boxes of each connected component of $D(w)$, i.e.,

$$\mathcal{E}ss(w) = \{(i, j) \in D(w) : (i + 1, j), (i, j + 1) \notin D(w)\}.$$

If it exists, the *accessible box* is the southmost then eastmost essential set box *not in the dominant component*. For example, if $w = 54278316 \in S_8$, $D(w)$ is depicted by:



Also, $\mathcal{E}ss(w) = \{(1, 4), (2, 3), (5, 3), (5, 6), (6, 1)\}$, and the accessible box is at $(5, 6)$. The *Lehmer code* of $w \in S_\infty$, denoted $\mathsf{code}(w)$ is the vector $(c_1, c_2, \ldots, c_L)$ where c_i equals the

[1] This is the "theorem of statistics" alluded to in [22].

number of boxes in row i of the Rothe diagram of w. We will assume L is minimum (i.e., $\mathsf{code}(w)$ does not have trailing zeros). By this convention, $\mathsf{code}(id) = ()$.

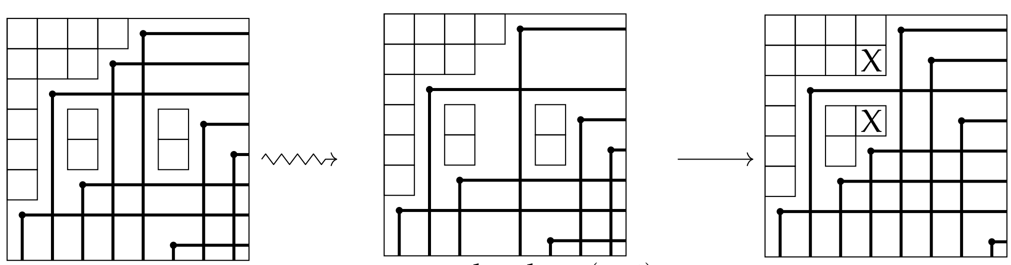
Fulton's criterion [17, Remark 9.17] states that u is vexillary if and only if there does not exist two essential set boxes where one is strictly northwest of the other. Thus, using the above picture of $D(w)$ we can see that w is not vexillary because of, e.g., $(1,4)$ and $(5,6)$.

2.2. **Description of $\mathcal{T}(w)$.** The original description of transition was given in [26]; this account is also given an exposition in [29, Sections 2.7.3, 2.7.4, 2.8.1]. We follow the graphical description given in [25] and its elaboration in [1]. There are some minor choices in describing the transition tree, and those of [25, 1] differ slightly from [26, 29].

We describe the graphical version of the transition algorithm to compute $\#\mathsf{Red}(w)$. The root of the tree is labelled by $D(w)$. If w is vexillary, stop. Otherwise, there exists an accessible box. (If not, $D(w)$ consists only of the dominant component and, by Fulton's criterion, w is vexillary, a contradiction.) The *pivots* of $D(w)$ are the maximally southeast $\bullet$'s of $G(w)$, say $b_1, b_2, \ldots, b_t$ that are northwest of the accessible box e.

If w is not vexillary, the children of w are defined as follows. For each $i = 1, 2, \ldots, t$, let R_i be the rectangle defined by b_i and e. Remove b_i and its rays from $G(w)$ to form $G^{(i)}(w)$. Order the boxes $\{v_i\}_{i=1}^m$ in English reading order. Move v_1 strictly north and strictly west to the closest position not occupied by another box of $D(w)$ or a ray from $G^{(i)}(w)$. Now, iterate this procedure with $v_2, v_3, \ldots$. At each step, v_j may move to a position vacated by earlier moves. The result is the diagram $D(w^{(i)})$ of some permutation $w^{(i)}$. These $D(w^{(i)})$'s are the children of $D(w)$. We call the transformation $D(w) \to D(w^{(i)})$ a *marching move*.

Example 2.1. Continuing our example, the pivots of w are $(1,5), (2,4)$ and $(3,2)$. We now obtain the child corresponding to the pivot $b_2 = (2,4)$:



w remove hook at $(2,4)$ $w^{(2)} = 56274318$

We have indicated by "X" the boxes that have moved. This process constructs one of the three children of w. In Figure 1 we draw the remainder of $\mathcal{T}(w)$. $\square$

If u is vexillary we define $\lambda(u)$ graphically by pushing all boxes of $D(u)$ northwest along the diagonal that it sits until a partition shape is reached; see [24, Section 3.2]. Concluding our running example, from Figure 1 we have

$$\begin{aligned}\#\mathsf{Red}(54278316) &= f^{\lambda(54672318)} + f^{\lambda(56274318)} + f^{\lambda(65342718)} + f^{\lambda(64532718)}\\ &= f^{4,3,3,3,1,1} + f^{4,4,3,2,1,1} + f^{5,4,2,2,1,1} + f^{5,3,3,2,1,1}\\ &= 730158.\end{aligned}$$

This result is a mild variation of [29, Proposition 2.8.1] (cf. [26]) using the marching moves. We make no claim of originality.

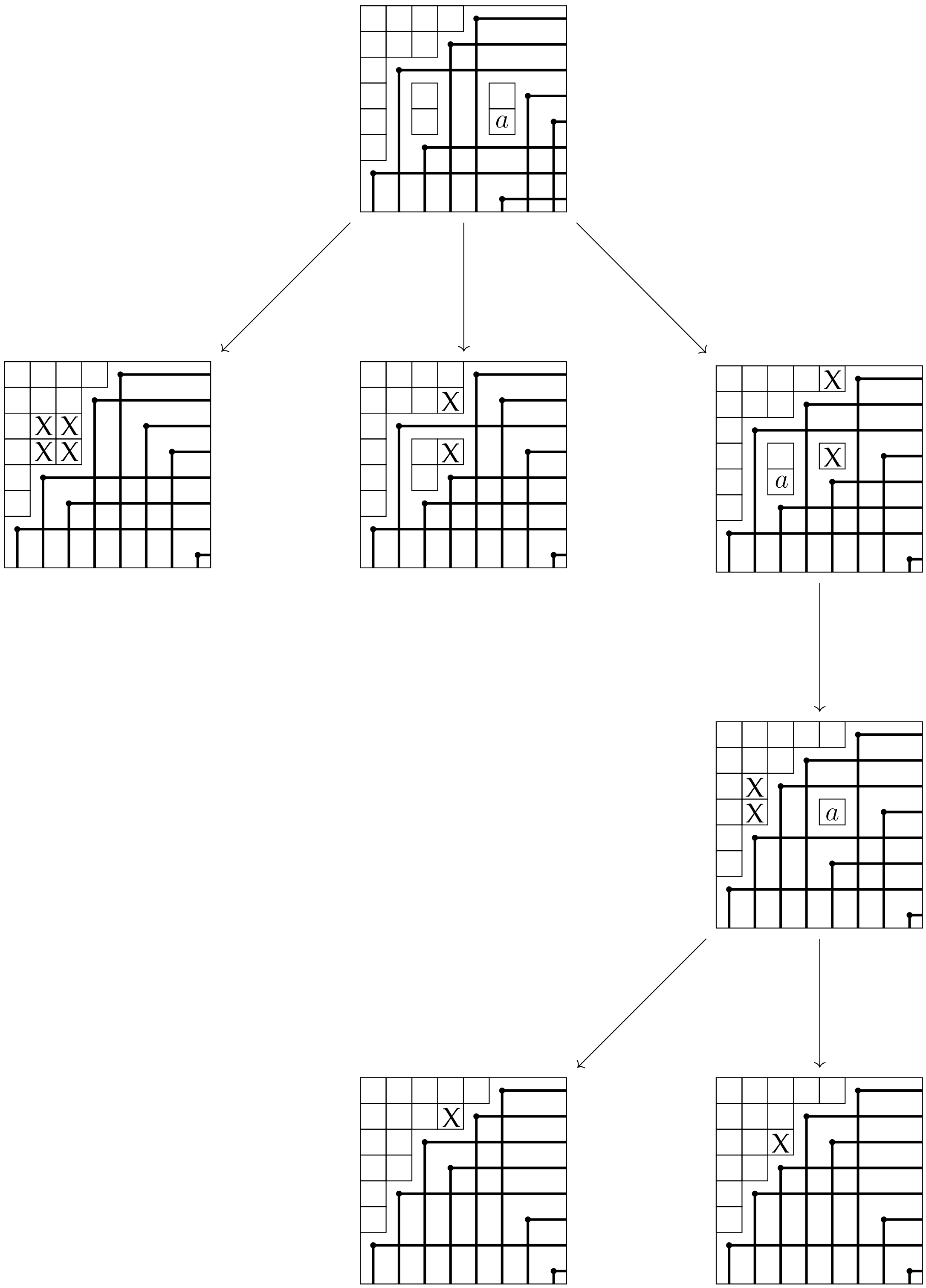


FIGURE 1. $\mathcal{T}(w)$ for $w = 54278316$. The a indicates the accessible box of each node. The X's describe which boxes of the parent moved. From this tree, we compute $\#\mathsf{Red}(w) = 730158$.

Theorem 2.2 (cf. [26, 29, 25])**.** $\#\mathsf{Red}(w) = \sum_{v \in \mathcal{L}(w)} f^{\lambda(v)}$.

Proof: We follow [1, Section 5.2], which elaborates on the notions from [25] in the case of Schubert polynomials $\mathfrak{S}_w$. We refer to [29, Chapter 2] for background.

Let (r, c) be the accessible box of $w \in S_n$ and set $k = w^{-1}(c)$. Also let $w' = w \cdot (r, k)$. Transition gives this recurrence for the *Schubert polynomials*:

$$\mathfrak{S}_w = x_r \mathfrak{S}_{w'} + \sum_{w''} \mathfrak{S}_{w''}, \tag{8}$$

where the summation is over the children w'' of w in $\mathcal{T}(w)$.

Let $1^N \times w \in S_{N+n}$ send $j \mapsto j$ for $1 \leq j \leq n$ and $j \mapsto w(j - N + 1) + N$ for $j \geq N$. Then

$$F_w = \lim_{N \to \infty} \mathfrak{S}_{1^N \times w} \in \mathbb{Z}[[x_1, x_2, \ldots]].^2$$

Moreover, since $w \in S_n$ then

$$F_w(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \mathfrak{S}_{1^n \times w}(x_1, x_2, \ldots, x_n, 0, 0, \ldots). \tag{9}$$

Now, by repeated application of (8) to $1^n \times w$,

$$\mathfrak{S}_{1^n \times w} = J(x_1, x_2, \ldots, x_{2n}) + \sum_{v \in \mathcal{L}(w)} \mathfrak{S}_{1^n \times v}, \tag{10}$$

where $J(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \equiv 0$.

Hence by setting $x_i = 0$ for $i > n$ in (10) we obtain, using (9) that

$$F_w(x_1, \ldots, x_n) = \sum_{v \in \mathcal{L}(w)} F_v(x_1, \ldots, x_n). \tag{11}$$

Let $s_\alpha(x_1, \ldots, x_n)$ be the Schur polynomial for a shape α. Since $v \in \mathcal{L}(w)$ is vexillary,

$$F_v(x_1, \ldots, x_n) = s_{\lambda(v)}(x_1, \ldots, x_n);$$

see, e.g., [29, Section 2.8.1]. Hence

$$F_w(x_1, \ldots, x_n) = \sum_{v \in \mathcal{L}(w)} s_{\lambda(v)}(x_1, \ldots, x_n). \tag{12}$$

We have that $[x_1 x_2 \cdots x_{\ell(w)}]F_w = \#\mathsf{Red}(w)$ and $[x_1 x_2 \cdots x_{\ell(w)}]s_{\lambda(v)}(x_1, \ldots, x_n) = f^{\lambda(v)}$. Now the result follows from these two facts combined with (12). □

3. Proof of Theorems 1.1 and 1.2

3.1. On the distribution of $\mathsf{EG}(w)$.

Lemma 3.1. *For any $w \in S_n$, $\mathsf{EG}(w) = \#\mathcal{L}(w)$.*

Proof. Combining results of [34, 13] gives

$$F_w(x_1, \ldots, x_{\ell(w)}) = \sum_{\lambda} a_{w,\lambda} s_\lambda(x_1, \ldots, x_{\ell(w)}) \tag{13}$$

where the sum is over partitions λ of size $\ell(w)$, and $a_{w,\lambda}$ is defined in Section 1.

2: In the conventions of [34], the limit is an expression for $F_{w^{-1}}$.

The Schur polynomials $s_\lambda(x_1, \ldots, x_{\ell(w)})$ for $|\lambda| = \ell(w)$ are a basis of the vector space $\Lambda_{\mathbb{Q}}^{(\ell(w))}[x_1, \ldots, x_{\ell(w)}]$ of degree $\ell(w)$ symmetric polynomials in $\{x_1, \ldots, x_{\ell(w)}\}$. Since (13) and (12) (where $n = \ell(w)$) are linear combinations for the same vector, we are done. $\square$

In view of Lemma 3.1, Theorems 1.1 and 1.2 are equivalent. It is easy to see that Theorem 1.1 follows from:

Theorem 3.2. *Fix $0 < \gamma < \frac{1}{2}$. There exists $\alpha > 0$ such that for n sufficiently large,*

$$\mathbb{P}(\mathsf{EG}(w) \geq 2^{\alpha n}) \geq 1 - \frac{1}{n^{2\gamma}}.$$

Proof of Theorem 3.2: Let $N_{\pi,n}(w)$ be the number of π patterns contained in $w \in S_n$.

Proposition 3.3. *Suppose in $\mathcal{T}(w)$ that the node u has exactly one child u'. then*

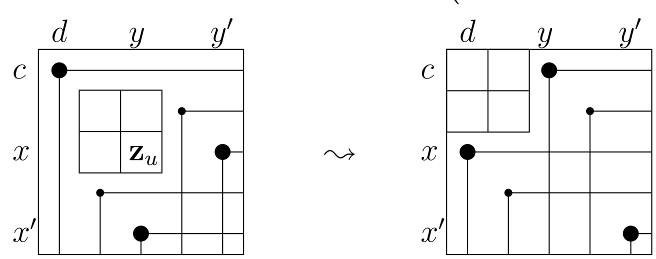
$$N_{2143,n}(u') \geq N_{2143,n}(u).$$

Proof of Proposition 3.3: Let the accessible box $\mathbf{z}_u$ of u be in position (x, y). By definition of $D(u)$, there is a $\bullet$ of $G(u)$ at $C = (x, y')$ for some $y' > y$, and there is a $\bullet$ at $B = (x', y)$ for some $x' > x$. Let b_1 be the unique pivot of $D(u)$, i.e., the southeastmost $\bullet$ that is northwest of $\mathbf{z}_u$ (as in Section 2.2). Suppose b_1 is at position $A = (c, d)$. Thus, $c < x$ and $d < y$.

By definition of the transition algorithm, all $\bullet$'s of $G(u)$ and $G(u')$ are in the same position, except A, B, C in $G(u)$ are respectively replaced by A', B', C' in $G(u')$ where

$$A = (c, d) \mapsto A' = (x, d)$$
$$B = (x', y) \mapsto B' = (c, y)$$
$$C = (x, y') \mapsto C' = (x', y')$$

Schematically, the march move looks as follows (we have thickened the moving $\bullet$'s).



Claim 3.4. *If there are two $\bullet$'s, other than $\{B, C\}$, that are weakly south and weakly east of $\mathbf{z}_u$ then one $\bullet$ must be (strictly) southeast of the other.*

Proof of Claim 3.4: Suppose not. Then let the two $\bullet$'s be at (q, r) and (m, n) where $q > m$ and $r < n$. Then there is a box $\mathbf{z} \neq \mathbf{z}_u$ of $D(u)$ in position (m, q), which is weakly south and weakly east of $\mathbf{z}_u$. Since $\mathbf{z}_u$ is not in the dominant component of $D(u)$, then $\mathbf{z}$ cannot be in that component either. Therefore, $\mathbf{z}_u$ is not the accessible box of $D(u)$, a contradiction. $\square$

Claim 3.5. *There is no $\bullet$ of $G(u)$ strictly north of row c and strictly between columns d and y. Similarly, there is no $\bullet$ of $G(u)$ strictly west of column d and strictly between rows c and x.*

Proof of Claim 3.5: We prove only the first sentence of the claim, as the second sentence is analogous. Suppose not; we may assume this $\bullet$ is maximally southeast with the assumed properties. Then $A = (c,d)$ and this $\bullet$ are two pivots for $D(u)$, which implies u has at least two children, contradicting the hypothesis of the Proposition. □

Let $\mathcal{F}_u$ consist of all embedding positions $i_1 < i_2 < i_3 < i_4$ of a 2143-pattern in u. Also, let $\mathcal{F}'_u$ be the subset of $\mathcal{F}_u$ consisting of those $i_1 < i_2 < i_3 < i_4$ such that

$$\{i_1, i_2, i_3, i_4\} \cap \{c, x, x'\} = \emptyset$$

(i.e., the positions do not involve the rows of A, B or C). Let

$$\mathcal{F}''_u = \mathcal{F}_u \setminus \mathcal{F}'_u.$$

Similarly, we define $\mathcal{F}_{u'}$, $\mathcal{F}'_{u'}$ and $\mathcal{F}''_{u'}$ in exactly the same way, except with respect to u'.

Since $\mathcal{F}'_u = \mathcal{F}'_{u'}$, it suffices to establish an injection

$$\psi : \mathcal{F}''_u \hookrightarrow \mathcal{F}''_{u'}.$$

In what follows, we will let $\bullet_i$ refer to the $\bullet$ in the diagram corresponding to the "i" in the 2143 pattern, for $1 \leq i \leq 4$. In addition, if i_1 is in the row of A we will write "$A = \bullet_2$", *etc.* We define now ψ in cases:

Case 1: ($B = \bullet_1$ or $B = \bullet_2$) The $\bullet_4$ and $\bullet_3$ appear strictly right of column y. This contradicts Claim 3.4. Hence, no elements of $\mathcal{F}''_u$ fall into this case.

Case 2: ($C = \bullet_1$ or $C = \bullet_2$) $\bullet_4$ and $\bullet_3$ appear strictly southeast of $\mathbf{z}_u$. As in Case 1, this contradicts Claim 3.4. Again, no elements of $\mathcal{F}''_u$ fall into this case.

Case 3: ($A = \bullet_1$) Let $\bullet_2$ be at position (r,s). Hence $r < c$ and $s > d$. If moreover, $s < y$ we contradict the first sentence of Claim 3.5. Hence, $s > y$. We must have that $\bullet_2 \notin \{A, B, C\}$ and $\bullet_4$ and $\bullet_3$ are strictly to the right of column y.

Subcase 3a: ($\bullet_4$ and $\bullet_3$ are both strictly south of row x) This contradicts Claim 3.4.

Subcase 3b: ($\bullet_4$ and $\bullet_3$ are both strictly north of row x) Thus $\{\bullet_3, \bullet_4\} \cap \{A, B, C\} = \emptyset$. The 2143 pattern $[\bullet_2, A, \bullet_4, \bullet_3]$ is destroyed by the marching move, i.e., $[\bullet_2, A', \bullet_4, \bullet_3]$ is not a 2143 pattern in u'. Now, in u' we now have the 2143 pattern $[\bullet_2, B', \bullet_4, \bullet_3]$. Hence we define

$$\psi([\bullet_2, A, \bullet_4, \bullet_3]) := [\bullet_2, B', \bullet_4, \bullet_3].$$

Subcase 3c: ($\bullet_4$ is strictly north of row x and $\bullet_3$ is strictly south of row x). Since $s > y$, $C \neq \bullet_3$. Hence $\{\bullet_3, \bullet_4\} \cap \{A, B, C\} = \emptyset$. The 2143 pattern $[\bullet_2, A, \bullet_4, \bullet_3]$ is destroyed by the marching move. However, in u' we now have the 2143 pattern $[\bullet_2, B', \bullet_4, \bullet_3]$. We again define

$$\psi([\bullet_2, A, \bullet_4, \bullet_3]) := [\bullet_2, B', \bullet_4, \bullet_3].$$

Subcase 3d: ($\bullet_3$ is in row x and $\bullet_4$ is strictly above row x) Then in fact $\bullet_3 = C$ while $\bullet_4 \notin \{A, B, C\}$. In this case,

$$\psi([\bullet_2, A, \bullet_4, C]) := [\bullet_2, B', \bullet_4, C'].$$

Subcase 3e: ($\bullet_4$ is in row x and $\bullet_3$ is strictly south of row x) Thus $\bullet_4 = C$ and $\bullet_3$ is strictly southeast of z_u. This contradicts Claim 3.4.

Case 4: ($A = \bullet_2$) Let the 1 be at position (r,s). Hence $r > c$ and $s < d$. If $r \leq x$ then we contradict the second sentence of Claim 3.5. Hence $r > x$. We have that $\bullet_4$ and $\bullet_3$ are in rows strictly south of x. Moreover, there must be a box e of $D(u)$ in the row of $\bullet_4$ and the column of $\bullet_3$ that is therefore strictly south of $\mathbf{z}_u$. Since the columns of $\bullet_4$ and $\bullet_3$ are strictly east of column d, the box e is not part of the dominant component of $D(u)$. Hence, $\mathbf{z}_u$ cannot be the accessible box, a contradiction. Thus, no elements of $\mathcal{F}''_u$ are in this case.

Case 5: ($A = \bullet_3$) Hence, in u, $\bullet_2, \bullet_1, \bullet_4$ are strictly north of row c. Thus $\{\bullet_1, \bullet_2, \bullet_4\} \cap \{A, B, C\} = \emptyset$ and $\bullet_2, \bullet_1, \bullet_4$ remain in the same place in u'. Set

$$\psi([\bullet_2, \bullet_1, \bullet_4, A]) := [\bullet_2, \bullet_1, \bullet_4, A'].$$

Case 6: ($A = \bullet_4$) $\bullet_3$ is strictly south of the row of A. If it is also weakly north of x, we contradict the second sentence of Claim 3.5. Hence $\bullet_3$ is strictly south of x, i.e., the row of e. Now, $\bullet_2, \bullet_1, \bullet_3$ are the same position in u and u' and $\{\bullet_1, \bullet_2, \bullet_3\} \cap \{A, B, C\} = \emptyset$. Here,

$$\psi([\bullet_2, \bullet_1, A, \bullet_3]) := [\bullet_2, \bullet_1, A', \bullet_3].$$

Since the row of A' is x the output is a 2143 pattern in u'.

Case 7: ($B = \bullet_4$) Let $\bullet_3$ be at (r, s). Thus $r > x'$ and $s < y$. There must be a box $e \in D(w)$ in position (x', s). Now, $\bullet_2$ and $\bullet_1$ are in columns strictly left of s and strictly above row r. Hence e cannot be in the dominant component of $D(w)$. Thus, since e is further south than $\mathbf{z}_u$, the latter is not accessible, a contradiction. So, no elements of $\mathcal{F}''_u$ appear in this case.

Case 8: ($B = \bullet_3$) Let $\bullet_4$ be in position (r, s).

Subcase 8a: ($r < c$) Therefore, $\bullet_1$ and $\bullet_2$ are also strictly above row c. Since $\bullet_1, \bullet_2$ and $\bullet_4$ stay in the same place in u and u' and B' is in row c in u'. Moreover, $\{\bullet_1, \bullet_2, \bullet_4\} \cap \{A, B, C\} = \emptyset$. We may define

$$\psi([\bullet_2, \bullet_1, \bullet_4, B]) := [\bullet_2, \bullet_1, \bullet_4, B'].$$

Subcase 8b: ($x < r < x'$) This contradicts Claim 3.4.

Subcase 8c: ($r = c$) This implies $A = \bullet_4$, which is impossible.

Subcase 8d: ($c < r < x$) We may assume $A \neq \bullet_1, \bullet_2$ since those cases are handled by Case 3 and Case 4. Now, $\bullet_1$ and $\bullet_2$ are strictly west of column y and strictly north of row x. By the assumption that $A = b_1$ is the (unique) pivot, combined with Claim 3.5, both $\bullet_1$ and $\bullet_2$ are strictly northwest of A. Thus, $\bullet_1$ and $\bullet_2$ are in the same place in u', and $\{\bullet_1, \bullet_2, \bullet_4\} \cap \{A, B, C\} = \emptyset$. Since A' is in row x, it make sense to let

$$\psi([\bullet_2, \bullet_1, \bullet_4, B]) := [\bullet_2, \bullet_1, \bullet_4, A'].$$

Subcase 8e: ($r = x$) Hence $C = \bullet_4$. For the same reasons as in Subcase 8d, both $\bullet_1$ and $\bullet_2$ are strictly northwest of A. Thus, $\bullet_1$ and $\bullet_2$ are in the same place in u' and $\{\bullet_1, \bullet_2\} \cap \{A, B, C\} = \emptyset$. In this case set

$$\psi([\bullet_2, \bullet_1, C, B]) := [\bullet_2, \bullet_1, B', A'].$$

Case 9: ($C = \bullet_3$) Let $\bullet_4$ be in position (r, s). Hence $s > y'$.

Subcase 9a: ($r < c$) Hence $\bullet_1, \bullet_2$ and $\bullet_4$ remain in the same place in u' and $\{\bullet_1, \bullet_2, \bullet_4\} \cap \{A, B, C\} = \emptyset$. Since C' is further south than C, we may set

$$\psi([\bullet_2, \bullet_1, \bullet_4, C]) := [\bullet_2, \bullet_1, \bullet_4, C'].$$

Subcase 9b: ($c < r < x$) We may also assume that $A \neq \bullet_1$ and $A \neq \bullet_2$, since those are handled in Case 3 and Case 4, respectively. Thus $\{\bullet_1, \bullet_2, \bullet_4\} \cap \{A, B, C\} = \emptyset$. Here,

$$\psi([\bullet_2, \bullet_1, \bullet_4, C]) := [\bullet_2, \bullet_1, \bullet_4, C'].$$

Subcase 9c: ($r = c$) Then $A = \bullet_4$, which is impossible.

Case 10: ($C = \bullet_4$) We may assume that $A \neq \bullet_1, \bullet_2$ (Case 3 and Case 4) and also $B \neq \bullet_3$ (Case 8). Therefore $\{\bullet_1, \bullet_2, \bullet_3\} \cap \{A, B, C\} = \emptyset$. Let $\bullet_3$ be in position (r, s).

Subcase 10a: ($y < s < y'$) This contradicts Claim 3.4.

Subcase 10b: ($s = y$) This means $\bullet_3 = B$, a situation we have ruled out/refer to Case 8.

Subcase 10c: ($s < y$) If moreover $r > x'$ then there exists $e \in D(w)$ in position (x', s), which is therefore strictly south of $\mathbf{z}_u$. Since column s is strictly east of the column of $\bullet_2$, e is not in the dominant component. Hence $\mathbf{z}_u$ is not accessible, a contradiction. Now $r \neq x'$ (since we assumed $B \neq \bullet_3$). Thus, $x < r < x'$ and it follows that $\bullet_3$ is in the same place in u'. By the reasoning of the first paragraph of Subcase 8d, $\bullet_1, \bullet_2$ are strictly northwest of A. Hence $\bullet_1, \bullet_2$ also remain in the same place in u'. Summing up, since B' is in row c, we may define

$$\psi([\bullet_2, \bullet_1, C, \bullet_3]) := [\bullet_2, \bullet_1, B', \bullet_3].$$

ψ is well-defined: The above cases handle each of the possibilities for A, B, C being one of $1, 2, 3, 4$. Our definition of ψ is shown to send an element of $\mathcal{F}''_u$ to an element of $\mathcal{F}''_{u'}$.

We also need that if an element of $\mathcal{F}''_u$ occurs in two cases, ψ sends them to the *same* element of $\mathcal{F}''_{u'}$. By inspection, the only overlapping situations are Subcase 3d$\leftrightarrow$Subcase 9b and Subcase 8d$\leftrightarrow$Case 10. In both these cases we define ψ to be consistent on the overlap.

ψ is an injection: This is by inspection of pairs of subcases where ψ's output was given. By our choice of notation, if $\bullet_i$ appears in the description of the input to ψ, it cannot be equal to A, B or C and hence in the output, it cannot be equal to A', B' or C' (as $\{A, B, C\}$ and $\{A', B', C'\}$ occupy the same rows). Therefore, if in two cases, some coordinate of the two outputs differ *symbolically*, those outputs cannot be equal. After ruling out these pairs, we are left with a few to check:

Subcase 3b, Subcase 3c: These differ in the fourth coordinate since in the former case, $\bullet_3$ is strictly north of row x and in the latter case, $\bullet_3$ is strictly south of row x.

Case 5 and Subcase 8d: These differ in the third coordinate since in the former case, $\bullet_4$ appears above row c whereas in the latter case, $\bullet_4$ is below row c.

Subcase 9a and Subcase 9b: These differ in the third coordinate for the same reason as the previous pair. □

Lemma 3.6. *Let $w \in S_n$ and suppose $u \to u'$ in $\mathcal{T}(w)$. Then*

$$N_{2143,n}(u) - N_{2143,n}(u') \leq 2n^3 + 3n^2 - n.$$

Proof of Lemma 3.6: Since $u \to u'$ in $\mathcal{T}(w)$, exactly three positions a, i, j differ between u and u'. We are claiming that

$$N_{2143,n}(u) - N_{2143,n}(u') \leq \binom{3}{3} 4 \binom{n}{1} + \binom{3}{2} 6 \binom{n}{2} + \binom{3}{1} 4 \binom{n}{3} = 2n^3 + 3n^2 - n. \tag{14}$$

Let $t_1 < t_2 < t_3 < t_4$ be the indices of a 2143-pattern in u. First suppose $\{t_1, t_2, t_3, t_4\} \cap \{a, i, j\} = \emptyset$. Clearly, $t_1 < t_2 < t_3 < t_4$ are indices of a 2143-pattern in u'. Therefore this case does not contribute to $N_{2143,n}(u) - N_{2143,n}(u')$.

Next assume $\#(\{t_1, t_2, t_3, t_4\} \cap \{a, i, j\}) = 1$. There are $\binom{3}{1}$ choices for which of a, i or j is in $\{t_1, t_2, t_3, t_4\}$. Then there are at most $\binom{n}{3}$ choices for $\{t_1, t_2, t_3, t_4\} \setminus \{a, i, j\}$. Finally there are 4 choices for which k satisfies $t_k \in \{a, i, j\}$. Therefore, this case contributes at most $\binom{3}{1}4\binom{n}{3}$ to $N_{2143,n}(u) - N_{2143,n}(u')$, thus explaining the third term of (14).

Similar arguments explain the first and second terms of (14) as the contributions to $N_{2143,n}(u) - N_{2143,n}(u')$ from the cases that

$$\#(\{t_1, t_2, t_3, t_4\} \cap \{a, i, j\}) = 3 \text{ and } \#(\{t_1, t_2, t_3, t_4\} \cap \{a, i, j\}) = 2,$$

respectively. The lemma thus follows. □

The following is known; see work of M. Bona [7] and of S. Janson, B. Nakamura, and D. Zeilberger [20]. The proof being not difficult, we include it for completeness.

Lemma 3.7. *For any $\pi \in S_k$, the expected number of occurrences of π as a pattern in $w \in S_n$ (selected using the uniform distribution) is $\binom{n}{k}/k!$.*

Proof of Lemma 3.7: For an increasing sequence $I = \{i_1 < i_2 < \ldots < i_k\}$ (in $[1, n]$), let

$$\mathsf{X}_I(w) = \begin{cases} 1 & \text{if } \pi \text{ is a pattern at the positions of } I; \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Thus, $N_{\pi,n} = \sum_I \mathsf{X}_I$. There are $\binom{n}{k}(n-k)!$ permutations such that I has pattern π. By linearity of expectation,

$$\mathbb{E}[N_{\pi,n}] = \sum_I \mathbb{E}[\mathsf{X}_I] = \binom{n}{k}^2 (n-k)!/n!,$$

and the lemma follows. □

Lemma 3.8. *Let $\mathcal{T}$ be a rooted tree with the property that along any path from the root to a leaf there are d nodes with at least two children. Then that tree has at least 2^d leaves.*

Proof of Lemma 3.8: Arbitrarily left-right order the descendants of the root of $\mathcal{T}$. After pruning, if necessary, we may assume each node as at most two children. Along any path from the root to a leaf, record "S" if a node has one child, and "L" if one steps to the left child and "R" if one goes to the right child. Thus, each leaf is uniquely encoded by an $\{S, L, R\}$ sequence. By hypothesis, each such sequence has at least d from $\{L, R\}$. Also, each of the 2^d-many $\{L, R\}$-sequences must be a subsequence of a unique leaf sequence. Hence there are at least 2^d leaves. □

By Chebyshev's inequality, for any $t \in \mathbb{R}_{>0}$, $\mathbb{P}(|N_{\pi,n} - \mu| \geq t\sigma) \leq 1/t^2$, and hence

$$\mathbb{P}(N_{\pi,n} \geq \mu - t\sigma) \geq 1 - 1/t^2.$$

For $\pi = 2143$, $\mu = \binom{n}{4}/4!$. Let $t = n^\gamma$ for the fixed choice $0 < \gamma < \frac{1}{2}$. Thus, we obtain

$$\mathbb{P}\left(\frac{N_{2143,n}}{2n^3 + 3n^2 + n} \geq \frac{\binom{n}{4}/4! - n^\gamma \sigma}{2n^3 + 3n^2 + n}\right) \geq 1 - \frac{1}{n^{2\gamma}}. \tag{16}$$

Define a random variable $Q : S_n \to \mathbb{Z}_{\geq 0}$ by

$$Q(w) = \min_{u \in \mathcal{L}(w)} \#\{v \text{ appears in a path from } w \text{ to } u \text{ in } \mathcal{T}(w) : \exists v' \neq v'', v \to v', v \to v''\}.$$

By Proposition 3.3 and Lemma 3.6,

$$Q(w) \geq \frac{N_{2143,n}(w)}{2n^3 + 3n^2 + n}. \tag{17}$$

Combining (16) and (17) gives

$$\mathbb{P}\left(Q \geq \frac{\binom{n}{4}/4! - n^{\gamma}\sigma}{2n^3+3n^2+n}\right) \geq 1 - \frac{1}{n^{2\gamma}}.$$

By [20, Section 4.1], the r-th central moment for $N_{\pi,n}$, i.e., $\mathbb{E}[(N_{\pi,n} - \mathbb{E}(N_{\pi,n}))^r]$, is a polynomial in n of degree $\lfloor r(k-\frac{1}{2})\rfloor$ where, recall, $\pi \in S_k$. Hence $\mathsf{Var}(N_{2143,n}) \in O(n^7)$ and $\sigma \in O(n^{3.5})$. Therefore there exists $\alpha > 0$ such that for n sufficiently large

$$\mathbb{P}(Q \geq \alpha n) \geq 1 - \frac{1}{n^{2\gamma}}. \tag{18}$$

Finally, by Lemma 3.1 and Lemma 3.8,

$$\mathsf{EG}(w) = \#\mathcal{L}(w) \geq 2^{Q(w)}. \tag{19}$$

The desired equality holds by (18) and (19) combined. □

3.2. **Remarks.** M. Bona [7] proves that the sequence of random variables

$$\widetilde{X_n} := \frac{N_{2143,n} - \mathbb{E}[N_{2143,n}]}{\sqrt{\mathsf{Var}(N_{2143,n})}}$$

is asymptotically normal, i.e., X_n converges in distribution to the standard normal variable $N(0,1)$. In particular, this means that for any $\epsilon > 0$, for any $a, b \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|\mathbb{P}(\widetilde{X_n} \in [a,b]) - \mathbb{P}(N(0,1) \in [a,b])| < \epsilon$. Thus we could use Bona's theorem to prove a more refined version of Theorem 3.2. However, since this does not change our basic conclusions (Theorems 1.1 and 1.2) we opted to state a result/proof that only appeals to Chebyshev's inequality.

In [5], $w \in S_n$ is defined to be *k-vexillary* if $\mathsf{EG}(w) = k$. I. G. Macdonald [28] proves that the proportion of vexillary permutations in S_n goes to zero as $n \to \infty$. Extending this, Theorem 3.2 implies:

Corollary 3.9. *Fix a positive integer k. Then $\lim_{n\to\infty} \mathbb{P}(w \in S_n$ is k-vexillary$) \to 0$.*

Using the relations

$$s_i s_j = s_j s_i \text{ for } |i-j| \geq 2, \text{ and } \; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{20}$$

one can transform between any two reduced words

$$s_{i_1} s_{i_2} \cdots s_{i_\ell} \iff s_{j_1} s_{j_2} \cdots s_{j_\ell} \in \mathsf{Red}(w);$$

see, e.g., [29, Proposition 2.1.6]. Hence, it follows that

$$\{i_1, i_2, \ldots, i_\ell\} = \{j_1, j_2, \ldots, j_\ell\}. \tag{21}$$

Let $\sigma^{(n)} = 214365 \cdots 2n\; 2n-1 \in S_{2n}$.

Proposition 3.10. $a_{\sigma^{(n)},\lambda} = f^{\lambda}$.

Proof. Fix any partition λ of size $2n-1$. Consider any row and column increasing filling T of λ, using each of the labels $\{1, 3, 5, \ldots, 2n-1\}$ precisely once. Let $\mathcal{A}_\lambda$ be the set of these tableaux. Also, let $\mathcal{B}_\lambda$ be the set of EG tableaux for the coefficient $a_{\sigma^{(n)},\lambda}$. $\mathsf{Red}(\sigma^{(n)})$ consists of all $n!$ rearrangements of the factors of $s_1 s_3 \cdots s_{2n-1}$. Hence, the column reading word of any $T \in \mathcal{A}_\lambda$ gives a reduced word for w. Thus, $\mathcal{A}_\lambda \subseteq \mathcal{B}_\lambda$. By (21), if $S \in \mathcal{B}_\lambda$, it must use each label of $\{1, 3, 5, \ldots, 2n-1\}$ exactly once. Since S must also be row and column increasing, we see $S \in \mathcal{A}_\lambda$. This gives $\mathcal{A}_\lambda = \mathcal{B}_\lambda$.

Given $T \in \mathcal{A}_\lambda (= \mathcal{B}_\lambda)$, let $\phi(T) \in \mathsf{SYT}(\lambda)$ be the standard Young tableau of shape λ obtained by sending label i in T to $\lceil \frac{i}{2} \rceil$. Clearly, $\phi : \mathcal{A}_\lambda \to \mathsf{SYT}(\lambda)$ is a bijection. Hence $a_{\sigma^{(n)},\lambda} = \#\mathcal{A}_\lambda = \#\mathsf{SYT}(\lambda) = f^\lambda$. □

Let $\mathsf{inv}(n)$ be the number of involutions of S_n. The following shows that the worst case and average case running time of transition is quite different:

Corollary 3.11. $\#\mathcal{L}(\sigma^{(n)}) = \mathsf{inv}(n) \sim \left(\frac{n}{e}\right)^{n/2} \frac{e^{\sqrt{n}}}{(4e)^{\frac{1}{4}}}$.

Proof. The equality holds since

$$\#\mathcal{L}(\sigma^{(n)}) = \mathsf{EG}(\sigma^{(n)}) = \sum_\lambda f^\lambda = \mathsf{inv}(n). \tag{22}$$

The first equality of (22) is Lemma 3.1, the second is Proposition 3.10 and the third is textbook (e.g., [35, Corollary 7.13.9]). The asymptotic statement is [23, Section 5.1.4]. □

The following conjecture has been proved by G. Orelowitz (private communication):

Conjecture 3.12. $a_{w,\lambda} \leq f^\lambda$.

We refer to his paper (in preparation) for application to the Edelman-Greene statistic.

4. Counting Hecke words

A sequence $(i_1, i_2, \ldots, i_N)$ is a *Hecke word* for $w \in S_n$ if $s_{i_1} \star s_{i_2} \star \cdots \star s_{i_N} = w$ where $\star$ is the *Demazure product* defined by

$$u \star s_i = \begin{cases} us_i & \text{if } \ell(us_i) = \ell(u) + 1 \\ u & \text{otherwise.} \end{cases}$$

Therefore, $N \geq \ell(w)$. Let $\mathsf{Hecke}(w, N)$ denote the set of Hecke words for w of length N.

4.1. Two generalizations of the Edelman-Greene formula (2).

We now give two formulas for computing $\mathsf{Hecke}(w, N)$. Both are known to experts, but we are unaware of any specific place that they appear in the literature.

Since $\mathsf{Hecke}(w, \ell(w)) = \mathsf{Red}(w)$, formula (23) below generalizes (2). Our second point is that in contrast with (5), even for vexillary permutations, (23) is not short.

Proposition 4.1. *There is a manifestly nonnegative combinatorial formula*

$$\#\mathsf{Hecke}(w, N) = \sum_{\lambda, |\lambda| = N} b_{w,\lambda} f^\lambda, \tag{23}$$

where $b_{w,\lambda}$ counts the number of row strictly increasing and column weakly increasing tableaux of shape λ whose top to bottom, right to left, column reading word is a Hecke word for w.

Let $M \geq 1$. There is a vexillary permutation $\pi \in S_{2M}$ with $\ell(\pi) = M^2$ such that

$$\#\{\lambda \in \mathsf{par}(M^2 + M) : b_{\pi,\lambda} > 0\} \geq \mathsf{par}(M),$$

where $\mathsf{par}(M)$ *is the number of partitions of size* M. *That is when* $w = \pi$ *and* $N = M^2 + M$, *(23) has at least* $\mathsf{par}(M)$*-many terms. Moreover,*

$$\sum_{\lambda:|\lambda|=M^2+M} b_{\pi,\lambda} \geq \mathsf{inv}(M).$$

Proof. We use the results of S. Fomin-A. N. Kirillov [15] who prove the following combinatorial formula for the *stable Grothendieck polynomial* G_w:

$$G_w = \sum_{(\mathbf{i},\mathbf{j})} (-1)^{\ell(w)-|\mathbf{j}|} \mathbf{x}^{\mathbf{j}},$$

where $\mathbf{i} = (i_1, \ldots, i_N) \in \mathsf{Hecke}(w, N)$, and $\mathbf{j} = (j_1 \leq j_2 \leq \cdots \leq j_N)$ are positive integers satisfying $j_t < j_{t+1}$ whenever $i_t \leq i_{t+1}$. This is a formal power series in $x_1, x_2, \ldots$.

For any $\mathbf{i} = \mathsf{Hecke}(w, N)$, the sequence $(1, 2, \ldots, N)$ can be used for $\mathbf{j}$. Hence,

$$(-1)^{N-\ell(w)} \#\mathsf{Hecke}(w, N) = [x_1 x_2 \cdots x_N] G_w. \tag{24}$$

S. Fomin-C. Greene [14, Theorem 1.2] states that, up to change of conventions,

$$G_w = \sum_{\lambda} (-1)^{|\lambda|-\ell(w)} b_{w,\lambda} s_\lambda. \tag{25}$$

Combining (24) and (25) gives (23).

C. Lenart [27] gave an expression for the *symmetric Grothendieck polynomial*:

$$G_\mu(x_1, x_2, \ldots, x_t) = \sum_{\lambda} (-1)^{|\lambda|-|\mu|} g_{\mu,\lambda} s_\lambda(x_1, \ldots, x_t) \tag{26}$$

where $\mu \subseteq \lambda \subseteq \widehat{\mu}$. Here $\widehat{\mu}$ is the unique maximal partition with t rows obtained by adding at most $i-1$ boxes to row i of μ for $2 \leq i \leq t$. In addition, $g_{\mu,\lambda}$ counts the number of *Lenart tableaux*, i.e., column and row strict tableaux of shape μ/λ with entries in the i-th row restricted to $1, 2, \ldots, i-1$ for each i.

Pick $\mu = M \times M$ and fix $t \geq M^2 + M$. Therefore $\widehat{\mu} = t \times M$. Hence, by (26),

$$(-1)^M [x_1 \cdots x_{M^2+M}] G_{M\times M}(x_1, \ldots, x_t) = \sum_{\lambda} g_{M\times M,\lambda} f^\lambda. \tag{27}$$

Here the sum is over $\mu \subseteq \lambda \subseteq \widehat{\mu}$ with $|\lambda| = M^2 + M$. Now, each such λ is of the form $(M \times M, \overline{\lambda})$ where $\overline{\lambda} \in \mathsf{par}(M)$ and is contained in $M \times M$. Notice that $g_{M\times M,\lambda} \geq f^{\overline{\lambda}}$, since for each such λ we can obtain a Lenart-tableau by filling the $\overline{\lambda}$ part with $1, 2, \ldots, M$ to obtain a standard tableau, in all possible ways. Hence, using [35, Corollary 7.13.9],

$$\sum_{\lambda \in \mathsf{Par}(M^2+M)} g_{M\times M,\lambda} \geq \sum_{\overline{\lambda}:|\overline{\lambda}|=M} f^{\overline{\lambda}} = \mathsf{inv}(M).$$

Finally, let $\pi = M+1, M+2, \ldots, 2M, 1, 2, 3 \ldots, M \in S_{2M}$. This is a vexillary permutation π with $\lambda(\pi) = M \times M$. By, e.g., [24, Lemma 5.4],

$$G_\pi(x_1, \ldots, x_t, 0, 0, \ldots,) = G_{M\times M}(x_1, \ldots, x_t, 0, 0, \ldots).$$

Since the Schur polynomials form a basis of the ring of symmetric polynomials, the right-hand sides of (26) and (25) coincide, i.e., $b_{\pi,\lambda} = g_{M\times M,\lambda}$ for every λ. The result follows. $\square$

Example 4.2. Let $w = 31524 = s_4 s_2 s_3 s_1$. Using (23) we obtain

$$\begin{aligned}\#\mathsf{Hecke}(w,5) &= \left(\begin{array}{ccc}1&2&4\\1&3&\end{array}, \begin{array}{ccc}1&2&4\\3&4&\end{array}\right) f^{3,2} + \left(\begin{array}{ccc}1&2&4\\3&&\\3&&\end{array}, \begin{array}{ccc}1&2&4\\1&&\\3&&\end{array}\right) f^{3,1,1} + \left(\begin{array}{cc}1&2\\3&4\\3&\end{array}, \begin{array}{cc}1&2\\1&4\\3&\end{array}\right) f^{2,2,1} \\ &= 2f^{3,2} + 2f^{3,1,1} + 2f^{2,2,1} = 32,\end{aligned}$$

which the reader may confirm by direct check. □

This next generalization of (2) is also manifestly nonnegative. It specializes in the vexillary case in a tantalizing way.

Proposition 4.3.

$$\#\mathsf{Hecke}(w,N) = \sum_{\lambda:\ell(w)\leq|\lambda|\leq N} c_{w,\lambda} f^{\lambda,N}, \tag{28}$$

where $c_{w,\lambda}$ is the number of row and column strict tableaux of shape λ whose top to bottom, right to left, column reading word is a Hecke word for w. If w is vexillary, then

$$\#\mathsf{Hecke}(w,N) = f^{\lambda(w),N}.$$

Proof. Work of A. Buch, A. Kresch, M. Shimozono, H. Tamvakis and the third author [10] proves that

$$G_w = \sum_{\lambda} (-1)^{\ell(w)-|\lambda|} c_{w,\lambda} G_\lambda \ \text{ where } \ G_\lambda = \sum_T (-1)^{|T|-|\lambda|} \mathbf{x}^T \tag{29}$$

and the latter sum is over all semistandard set-valued tableaux of shape λ [8]. Therefore by (24) we have

$$\begin{aligned}(-1)^{N-\ell(w)}\#\mathsf{Hecke}(w,N) &= [x_1 x_2 \cdots x_N] G_w \\ &= [x_1 x_2 \cdots x_N] \sum_\lambda (-1)^{\ell(w)-|\lambda|} c_{w,\lambda} G_\lambda \\ &= \sum_{\lambda:|\lambda|\leq N} (-1)^{\ell(w)-|\lambda|} c_{w,\lambda} [x_1 x_2 \cdots x_N] G_\lambda \\ &= \sum_{\lambda:|\lambda|\leq N} (-1)^{\ell(w)-|\lambda|} c_{w,\lambda} (-1)^{N-|\lambda|} f^{\lambda,N} \\ &= \sum_{\lambda:|\lambda|\leq N} (-1)^{N+\ell(w)} c_{w,\lambda} f^{\lambda,N},\end{aligned}$$

proving (28). For the second statement, by [32, Lemma 5.4], when w is vexillary then $G_w = G_\lambda$, and the above sequence of equalities simplifies, as desired. □

Example 4.4. Again let $w = 31524$ as in Example 4.2. Now applying (28) gives

$$\begin{aligned}\#\mathsf{Hecke}(w,5) &= \left(\begin{array}{cc}1&2\\3&4\end{array}\right) f^{(2,2),5} + \left(\begin{array}{ccc}1&2&4\\3&&\end{array}\right) f^{(3,1),5} + \left(\begin{array}{ccc}1&2&4\\3&4&\end{array}\right) f^{(3,2),5} \\ &= 10 + 17 + 5 = 32,\end{aligned}$$

in agreement with Example 4.2. One can check the $f^{\lambda,N}$ computations either directly, or by using

$$f^{\lambda,N} = [x_1 \cdots x_N] G_\lambda \tag{30}$$

combined with (26). $\square$

Proposition 4.3 is our central motivation for Problem 1.4.

Proposition 4.5. *Fix k. There is an $|\mu|^{O(1)}$ algorithm to compute $f^{\mu,N}$ where $N \leq |\mu| + k$.*

Proof. We use (30) combined with (26) and describe the possible Lenart tableaux. First, we look for $\mu \subseteq \lambda \subseteq \overline{\mu}$ where $|\lambda| = |\mu| + k$. Such λ correspond to a choice of k rows $r_1 \leq r_2 \leq \ldots \leq r_k$ to add a box, among $\ell(\mu) + k - 1$ choices (rows $2, 3, \ldots, \ell(\mu) + k$). There are $\binom{\ell(\mu)+2k-2}{k} \in |\mu|^{O(1)}$ many ways to do this. For each such choice, it takes constant time to verify that λ is a partition. For those cases, we construct a possible Lenart tableau T by filling row r_i in at most $(r_i - 1)^k$ ways. Since $r_i \leq \ell(\mu) + k$, there are $|\mu|^{O(1)}$-many possible row strictly increasing tableaux T. It remains to determine if T is actually a Lenart tableau, which takes constant time (since k is fixed). Finally, to each tableau, we must compute f^λ via (3). This takes $|\lambda|^{O(1)}$-time. Now, since $|\lambda| = |\mu| + k$ and k is fixed, it also takes $|\mu|^{O(1)}$-time. Moreover, $\log(f^\lambda) \leq \log |\lambda|! \in O(|\mu| \log |\mu|)$. Hence, summing the at most $\binom{\ell(\mu)+2k-2}{k}$ hook-length calculations, also takes $|\mu|^{O(1)}$-time, as desired. $\square$

Example 4.6. We elaborate on the proof of Theorem 4.5 in the case $k = 2$. Here we look for $\mu \subseteq \lambda \subseteq \overline{\mu}$ where $|\lambda| = |\mu| + 2$. Such λ correspond to a choice of two rows $r_1 \leq r_2$ to add a box, among $\ell(\mu) + 1$ choices (rows $2, 3, \ldots, \ell(\mu) + 2$). If $r_1 = r_2$, $g_{\mu,\lambda} = \binom{r_2-1}{2}$. Otherwise if $r_1 < r_2$, there are $\binom{\ell(\mu)+1}{2}$ many choices. Assuming λ is a partition, there are two cases. If the two boxes are in different columns $g_{\mu,\lambda} = (r_1 - 1)(r_2 - 1)$. Otherwise, if they are in the same column (and hence $r_2 = r_1 + 1$), then $g_{\mu,\lambda} = \binom{r_2-1}{2}$. Now apply (26).

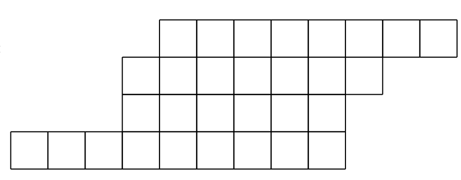
For $\lambda = \delta_{100} = (100, 99, \ldots, 3, 2, 1)$ and $N = \binom{100}{2} + 2$, this procedure exactly computes $f^{\lambda,N} = \#\mathsf{Hecke}(w_0, N) = 3.75\ldots \times 10^{7981}$. $\square$

4.2. **Application to Euler characteristics of Brill-Noether varieties (after [2, 11]).** Counting standard set-valued tableaux has been given geometric impetus through work of [2, 11] on *Brill-Noether varieties*. More precisely, following [11, Definition 1.2], let $g, r, d \in \mathbb{Z}_{\geq 0}$. Suppose $\alpha = (\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r)$ and $\beta = (\beta_0 \leq \beta_1 \leq \cdots \leq \beta_r)$ be sequences in $\mathbb{Z}^{r+1}_{\geq 0}$. Let $\mathsf{CP} = \mathsf{CP}(g, r, d, \alpha, \beta)$ be the skew Young diagram with boxes

$$\{(x, y) \in \mathbb{Z}^2 \ : \ 0 \leq y \leq r, -\alpha_y \leq x < g - d + r + \beta_{r-y}\}.$$

Example 4.7. If $g = 45, d = 43, r = 3, \alpha = (0, 1, 1, 4), \beta = (0, 0, 1, 3)$ then

$\mathsf{CP} =$



$\square$

Let $\chi(G^{r,\alpha,\beta}_d(X, p, q))$ be the algebraic Euler characteristic of the Brill-Noether variety $G^{r,\alpha,\beta}_d(X, p, q)$.

Theorem 4.8 (M. Chan-N. Pfleuger [11]). $(-1)^{g-|\mathsf{CP}|}\chi(G_d^{r,\alpha,\beta}(X,p,q)) = f^{\mathsf{CP},g}$.

Given λ/μ, construct a (321-avoiding) permutation $w_{\lambda/\mu}$ by filling all boxes in the same northwest-southeast diagonal with the same entry, starting with 1 on the northeastmost diagonal and increasing consecutively as one moves southwest. Call this filling $T_{\lambda/\mu}$. Let $(r_1, r_2, \ldots, r_{|\lambda/\mu|})$ be the left-to-right, top-to-bottom, row reading word of $T_{\lambda/\mu}$. Define $w_{\lambda/\mu} = s_{r_1}s_{r_2}\cdots s_{r_{|\lambda/\mu|}}$.

Example 4.9. If $\lambda/\mu = (12,10,9,9)/(4,3,3,0)$ be CP from Example 4.7. Then

$T_{\lambda/\mu} =$

				8	7	6	5	4	3	2	1
			10	9	8	7	6	5	4		
			11	10	9	8	7	6			
15	14	13	12	11	10	9	8	7			

The reading word is

$$(8,7,6,5,4,3,2,1,10,9,8,7,6,5,4,11,10,9,8,7,6,15,14,13,12,11,10,9,8,7)$$

and $w_{\lambda/\mu} = 9,1,2,11,3,12,16,4,5,6,7,8,10,13,14,15 \in S_{16}$. ☐

Earlier, in [2], a *cancellative* combinatorial formula for $\chi(G_d^{r,\alpha,\beta}(X,p,q))$ is given. In [11], another cancellative formula is given, as a signed sum involving counts of (ordinary) skew standard Young tableaux. See [11, Theorem 6.6] (and the discussion of [14] in [11, Section 6]) as well as [2, Theorem A, Theorem C].

Proposition 4.10. *Let w_{CP} is a 321-avoiding permutation defined as above. Then:*

$$(-1)^{g-|\mathsf{CP}|}\chi(G_d^{r,\alpha,\beta}(X,p,q)) = \sum_{\lambda,|\lambda|=g} b_{w_{\mathsf{CP}},\lambda} f^{\lambda},$$

and

$$(-1)^{g-|\mathsf{CP}|}\chi(G_d^{r,\alpha,\beta}(X,p,q)) = \sum_{\lambda:|CP|\leq|\lambda|\leq g} c_{w_{\mathsf{CP}},\lambda} f^{\lambda,g}.$$

Proof. By [8, Section 2], $G_{w_{\nu/\mu}} = G_{\nu/\mu} := \sum_T (-1)^{|T|-|\nu/\mu|}\mathbf{x}^T$, where the sum is over semi-standard set-valued tableaux of skew shape ν/μ. Therefore if $f^{\nu/\mu,N}$ is the number of standard set-valued tableaux of this shape with N entries then

$$f^{\nu/\mu,N} = \#\mathsf{Hecke}(w_{\nu/\mu}, N). \tag{31}$$

Now combine this with Theorem 4.8, Proposition 4.1 and 4.3. ☐

The second formula expresses $(-1)^{g-|\mathsf{CP}|}\chi(G_d^{r,\alpha,\beta}(X,p,q))$ as a cancellation-free sum of Euler characteristics of other Brill-Noether varieties. *Is there a geometric explanation of this?*

5. Three importance sampling algorithms

5.1. **Estimating $\#\mathsf{Red}(w)$.** Define a random variable Y_w for $w \in S_n$, as follows:

---

if w is vexillary then
  $\mathsf{Y}_w = f^{\lambda(w)}$
else

$C = \{w' \text{ is a child of } w \text{ in } \mathcal{T}(w)\}$
Choose $W' \in C$ uniformly at random
$\mathsf{Y}_w = \#C \times \mathsf{Y}_{w'}$

Proposition 5.1. *Let* $w \in S_n$. *Then* $\mathbb{E}(\mathsf{Y}_w) = \#\mathsf{Red}(w)$.

Proof. We induct on $h = h(w) \geq 0$, the height of $\mathcal{T}(w)$, i.e., the maximum length of any path from the root to a leaf. In the base case, $h = 0$, w is vexillary and thus, by (5),

$$\mathbb{E}(\mathsf{Y}_w) = f^{\lambda(w)} = \#\mathsf{Red}(w).$$

Our induction hypothesis is that $\mathbb{E}(\mathsf{Y}_u) = \#\mathsf{Red}(u)$ whenever $h(u) < h(w)$. Now

$$\begin{aligned}\mathbb{E}(\mathsf{Y}_w) &= \sum_{w' \in C} \mathbb{E}(\mathsf{Y}_w | W' = w')\mathbb{P}(W' = w') \\ &= \frac{1}{\#C} \sum_{w' \in C} \mathbb{E}(\mathsf{Y}_w | W' = w') \\ &= \frac{1}{\#C} \sum_{w' \in C} \mathbb{E}(\#C \times \mathsf{Y}_{w'}) \\ &= \sum_{w' \in C} \mathbb{E}(\mathsf{Y}_{w'}) \\ &= \sum_{w' \in C} \#\mathsf{Red}(w') \ \ \text{(induction hypothesis)} \\ &= \#\mathsf{Red}(w).\end{aligned}$$

The last equality is by construction of the transition algorithm and Theorem 2.2. ☐

Example 5.2. Let $w = 43817625 \in S_8$. We have the following sequence of transition steps

$$43817625 \xrightarrow{3} 53817426 \xrightarrow{1} 53827146 \xrightarrow{3} 63825147 \xrightarrow{2} 63842157 \xrightarrow{2} 73642158.$$

The number of children is indicated at each stage. The final permutation is vexillary, and $f^{\lambda(73642158)} = f^{6,4,2,2,1} = 243243$. Hence one sample is $3 \times 1 \times 3 \times 2 \times 2 \times 243243 = 8756748$. Using 10^3 samples gives an estimate of $2.1(\pm 0.1) \times 10^6$, versus $\#\mathsf{Red}(w) = 2085655$.[3] ☐

Example 5.3 ($w = \sigma^{(n)} = 2143 \cdots 2n\ 2n-1$). When $n = 10$ (so $\sigma^{(n)} \in S_{20}$), 10^5 samples estimates $3.6(\pm 0.1) \times 10^6$, which is close to the exact value $10! = 3628800$. When $n = 30$ ($\sigma^{(n)} \in S_{60}$), 10^6 samples suggests $2.4(\pm 1.1) \times 10^{32}$ whereas $30! = 2.65 \ldots \times 10^{32}$. ☐

Example 5.4 (Estimating the number of skew standard Young tableaux). We continue Example 4.9. Let $f^{\lambda/\mu}$ be the number of standard Young tableaux of shape λ/μ. By a result of S. Billey-W. Jockusch-R. P. Stanley [4, Corollary 2.4], $F_{w_{\lambda/\mu}} = s_{\lambda/\mu}$. Taking the coefficient of $x_1 x_2 \cdots x_{|\lambda/\mu|}$ on both sides implies $\#\mathsf{Red}(w_{\lambda/\mu}) = f^{\lambda/\mu}$. One has the textbook determinantal formula

$$f^{\lambda/\mu} = |\lambda/\mu|! \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)_{i,j=1}^{t}. \tag{32}$$

So $f^{\lambda/\mu} = 73064598262110 \approx 7.31 \times 10^{13}$. A 10^4 sample estimate is $7.3(\pm 0.1) \times 10^{13}$. ☐

[3] Code available at `https://github.com/ICLUE/reduced-word-enumeration`

5.2. **Estimating** $\#\mathsf{Hecke}(w, N)$**.** We propose a different importance sampling algorithm, to compute $\#\mathsf{Hecke}(w, N)$. For $N < \ell(w)$ the random variable $\mathsf{Z}_{w,N}$ is equal to 0 and for $N \geq \ell(w)$, it is recursively defined by:

---

if $w = id$ then
    if $N = 0$ then $\mathsf{Z}_{w,N} = 1$ else $\mathsf{Z}_{w,N} = 0$
else
    $D = \{i : w(i) > w(i+1)\}$
    Choose $I \in D$ and $\theta \in \{0, 1\}$ independently and uniformly at random
    if $\theta = 0$ then $\mathsf{Z}_{w,N} = 2\#D \times \mathsf{Z}_{w,N-1}$ else $\mathsf{Z}_{w,N} = 2\#D \times \mathsf{Z}_{ws_I,N-1}$

Proposition 5.5. *Let $w \in S_n$ and $N \geq \ell(w)$. Then $\mathbb{E}(\mathsf{Z}_{w,N}) = \#\mathsf{Hecke}(w, N)$.*

Proof. First we claim
(33)
$$\#\mathsf{Hecke}(w, N) = \begin{cases} 1 & \text{if } w = id \text{ and } N = 0 \\ 0 & \text{if } w = id \text{ and } N > 0 \\ \sum_{i \in D} (\#\mathsf{Hecke}(ws_i, N-1) + \#\mathsf{Hecke}(w, N-1)) & \text{otherwise.} \end{cases}$$
The unique Hecke word for $w = id$ is the empty word; this explains the first two cases.

Thus assume $w \neq id$ and $N \geq \ell(w)$. Suppose that $(i_1, i_2, \ldots, i_N) \in \mathsf{Hecke}(w, N)$.

Claim 5.6. *i_N is the position of a descent of w, i.e., $w(i_N) > w(i_N + 1)$.*

Proof of Claim 5.6: Consider $w' := s_{i_1} \star s_{i_2} \star \cdots \star s_{i_{N-1}}$. Either $\ell(w') = \ell(w)$ or $\ell(w') = \ell(w) - 1$. In the former case then if i_N is the position of an ascent of $w' = w$ then $w = w' \star s_{i_N}$ would create a descent at that position, a contradiction. In the latter case, w' had an ascent at position i_N which becomes a descent in $w' \star s_{i_N} = w' s_{i_N}$. ☐

Claim 5.6 implies the existence of a bijection

$$\mathsf{Hecke}(w, N) \xrightarrow{\sim} \left(\bigcup_{i \in D} \mathsf{Hecke}(ws_i, N-1) \times \{i\} \right) \cup \left(\bigcup_{i \in D} \mathsf{Hecke}(w, N-1) \times \{i\} \right), \tag{34}$$

defined by $(i_1, i_2, \ldots, i_{N-1}, i_N) \in \mathsf{Hecke}(w, N) \mapsto ((i_1, i_2, \ldots, i_{N-1}), i_N)$.[4] Therefore, by taking cardinalities on both sides of (34) we obtain the third case of (33).

Returning to proposition itself, we induct on $N \geq 0$. The case $N = 0$ holds by the first case of (33) and the definition $Z_{w,N} = 0$ if $N < \ell(w)$. For $N > 0$,

$$\begin{aligned} \mathbb{E}(\mathsf{Z}_{w,N}) &= \sum_{i \in D} \mathbb{E}(\mathsf{Z}_{w,N} | I = i, \theta = 0) \mathbb{P}(I = i) \mathbb{P}(\theta = 0) \\ &\quad + \sum_{i \in D} \mathbb{E}(\mathsf{Z}_{w,N} | I = i, \theta = 1) \mathbb{P}(I = i) \mathbb{P}(\theta = 1) \\ &= \sum_{i \in D} \mathbb{E}(2\#D \times \mathsf{Z}_{w,N-1}) \frac{1}{\#D} \times \frac{1}{2} + \sum_{i \in D} \mathbb{E}(2\#D \times \mathsf{Z}_{ws_i,N-1}) \frac{1}{\#D} \times \frac{1}{2} \end{aligned}$$

[4] If $N = \ell(w)$, then $\mathsf{Hecke}(w, N) = \mathsf{Red}(w)$ and $\mathsf{Hecke}(w, N-1) = \emptyset$. In this case, (34) reduces to the bijection $\mathsf{Red}(w) \xrightarrow{\sim} \bigcup_{i \in D} \mathsf{Red}(ws_i) \times \{i\}$.

$$= \sum_{i \in D} (\mathbb{E}(\mathsf{Z}_{w,N-1}) + \mathbb{E}(\mathsf{Z}_{ws_i,N-1}))$$
$$= \sum_{i \in D} (\#\mathsf{Hecke}(w, N-1) + \#\mathsf{Hecke}(ws_i, N-1))$$
$$= \#\mathsf{Hecke}(w, N),$$

where we have applied induction (on N) and the third case of (33). □

Example 5.7. One can explicitly generate all 2030964 elements of $\mathsf{Hecke}(351624, 13)$. Using 2000 samples gives the estimate $2.0(\pm 0.2) \times 10^6$. □

Example 5.8. By [32, Corollary 1.3],

$$\#\mathsf{Hecke}\left(w_0, \binom{n}{2} + 1\right) = \frac{\binom{n}{2}\left[\binom{n}{2}+1\right]}{n} \times \#\mathsf{Red}(w_0). \tag{35}$$

For $n = 10$, $\#\mathsf{Hecke}(w_0, 46) = 5.65\ldots \times 10^{28}$. Using 10^7 samples we obtained an unimpressive estimate of $\approx 9.0(\pm 6.4) \times 10^{28}$. □

The Z-algorithm restricts to an algorithm to compute $\#\mathsf{Red}(w)$. However, the Y-algorithm of Subsection 5.1 sometimes has better convergence in this case. This suggests a "hybrid" algorithm. Define $\mathsf{H}_{w,N}$ to be 0 if $N < \ell(w)$. Otherwise,

---

if $N = \ell(w)$ then $\mathsf{H}_{w,N} = Y_w$
else if $w = id$ then
  if $N = 0$ then $\mathsf{H}_{w,N} = 1$ else $\mathsf{H}_{w,N} = 0$
else
  $D = \{i : w(i) > w(i+1)\}$
  Choose $I \in D$ and $\theta \in \{0, 1\}$ independently and uniformly at random
  if $\theta = 0$ then $\mathsf{H}_{w,N} = 2\#D \times \mathsf{H}_{w,N-1}$ else $\mathsf{H}_{w,N} = 2\#D \times \mathsf{H}_{ws_I,N-1}$

Proposition 5.9. *Let $w \in S_n$. Then $\mathbb{E}[\mathsf{H}_{w,N}] = \#\mathsf{Hecke}(w, N)$.*

We omit the proof, as it is a straightforward modification of the argument for Proposition 5.5, using Proposition 5.1.

Example 5.10. Let $w = 361824795 \in S_9$; hence $\ell(w) = 12$. Using 10^6 samples with either the Z or H algorithm, it seems $\#\mathsf{Hecke}(w, 25) \approx 6.0(\pm 0.1) \times 10^{16}$. For Example 5.8, with 10^7 samples, the H algorithm gives a better estimate for $\#\mathsf{Hecke}(w_0, 46)$ of $4.6(\pm 1.1) \times 10^{28}$. □

Example 5.11. We use Proposition 4.5 to compute $\#\mathsf{Hecke}(w_0, \binom{n}{2} + 2)$. When $n = 7$, $\#\mathsf{Hecke}(w_0, 23) = 2.54\ldots \times 10^{12}$. Using 10^6 samples indicates $\approx 2.4(\pm 0.2) \times 10^{12}$. For $n = 10$, $\#\mathsf{Hecke}(w_0, 47) = 6.01\ldots \times 10^{30}$. A 10^7 sample estimate is $\approx 4.6(\pm 1.5) \times 10^{30}$. □

Example 5.12 (Skew set-valued tableaux). To estimate $f^{\lambda/\mu,N}$ for $\lambda/\mu = (12, 10, 9, 9)/(4, 3, 3, 0)$ and $N = 45$, we applied (31) and the Z-algorithm with 10^7 samples to predict $f^{\lambda/\mu,45} = \#\mathsf{Hecke}(w_{\lambda/\mu}, 45) \approx 1.3(\pm 0.1) \times 10^{33}$. This is backed by the same estimate using the H-algorithm with 10^6 samples. We have thus estimated the value of $(-1)^{g-|\mathsf{CP}|}\chi(G_d^{r,\alpha,\beta}(X, p, q))$ for the parameters of Example 4.7. There are a number of different ways to theoretically compute this value ([2], [11], Proposition 4.10). *What is the exact value?* □

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