NEWELL-LITTLEWOOD NUMBERS II: EXTENDED HORN INEQUALITIES

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ABSTRACT. The Newell-Littlewood numbers $N_{\mu,\nu,\lambda}$ are tensor product multiplicities of Weyl modules for classical Lie groups, in the stable limit. For which triples of partitions (μ, ν, λ) does $N_{\mu,\nu,\lambda} > 0$ hold? The Littlewood-Richardson coefficient case is solved by the Horn inequalities (in work of A. Klyachko and A. Knutson-T. Tao). We extend these celebrated linear inequalities to a much larger family, suggesting a general solution.

1. INTRODUCTION

1.1. **Background.** This is a sequel to [6]. We study *Newell-Littlewood numbers* [14, 13]

$$(1) \qquad N_{\mu,\nu,\lambda} = \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta}^{\mu} c_{\alpha,\gamma}^{\nu} c_{\beta,\gamma}^{\lambda};$$

the indices are partitions in $\mathrm{Par}_n = \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$. Also, $c_{\alpha,\beta}^{\mu}$ is the *Littlewood-Richardson coefficient*. The Newell-Littlewood numbers are tensor product multiplicities for the irreducible representations of a classical Lie algebra $\mathfrak{g}$ in the "stable limit"; we refer the reader to [6] for additional background and references, such as to [7].

Consider the problem:

Classify $(\mu, \nu, \lambda) \in \mathrm{Par}_n^3$ *such that* $N_{\mu,\nu,\lambda} > 0$.

Since $N_{\mu,\nu,\lambda} = c_{\mu,\nu}^{\lambda}$ if $|\lambda| = |\mu| + |\nu|$ [6, Lemma 2.2(II)], a subproblem asks when $c_{\mu,\nu}^{\lambda} > 0$? The solution to that case is 1990's combined breakthrough work of A. Klyachko [9] and A. Knutson-T. Tao [10]. For $I = \{i_1 < \cdots < i_d\} \subseteq \mathbb{Z}_{>0}$, let

$$\tau(I) := (i_d - d \geq \cdots \geq i_2 - 2 \geq i_1 - 1) \in \mathrm{Par}_d.$$

Theorem 1.1. *([9], [10]) Let* $\mu, \nu, \lambda \in \mathrm{Par}_n$ *such that* $|\lambda| = |\mu| + |\nu|$. *Then* $c_{\mu,\nu}^{\lambda} > 0$ *if and only if for every* $d < n$, *and every triple of subsets* $I, J, K \subseteq [n]$ *of cardinality* d *such that* $c_{\tau(I),\tau(J)}^{\tau(K)} > 0$,

$$(2) \qquad \sum_{k \in K} \lambda_k \leq \sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j.$$

The recursive *Horn inequalities* (2) were introduced in A. Horn's 1962 paper [8]. The inequalities have a pre-history [5, 3].

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and Λ_+ be the set of dominant integral weights. Suppose V_λ is the irreducible representation of $\mathfrak{g}$ indexed by $\lambda \in \Lambda_+$. Define multiplicities $m_{\mu,\nu}^{\lambda}$ by

$$V_\mu \otimes V_\nu = \bigoplus_{\lambda \in \Lambda_+} V_\lambda^{\oplus m_{\mu,\nu}^{\lambda}}.$$

The *tensor semigroup* is

$$\mathsf{Tensor}(\mathfrak{g}) = \{(\mu, \nu, \lambda) \in \Lambda_+^3 : m_{\mu,\nu}^\lambda > 0\}.$$

Compare this with the *saturated tensor cone*,

$$\mathsf{SatTensor}(\mathfrak{g}) = \{(\mu, \nu, \lambda) \in \Lambda_+^3 : \exists t \in \mathbb{Z}_{>0}, m_{t\mu,t\nu}^{t\lambda} > 0\}.$$

There are generalized Horn inequalities describing $\mathsf{SatTensor}(\mathfrak{g})$ [2]. In type A, the *saturation theorem* [10] implies

$$\mathsf{Tensor}(\mathfrak{sl}(n)) = \mathsf{SatTensor}(\mathfrak{sl}(n)).$$

For the other classical types, this is either false, or not known (see [16, 12]). Therefore, although each $N_{\mu,\nu,\lambda}$ is a tensor product multiplicity for $\mathfrak{g}$ of classical type, these results, while definitely related, do not immediately solve the problem.[1]

N. Ressayre [15] introduces different generalized Horn inequalities that hold when the *Kronecker coefficient* $g_{\mu,\nu,\lambda}$ is nonzero. Those coefficients are also tensor product multiplicities, but for Specht modules, not Weyl modules.

1.2. **Main results.** We suggest an answer to our problem, by introducing a large, new family of inequalities extending (2).

Definition 1.2. *An extended Horn inequality is*

$$(3) \qquad 0 \leq \sum_{i \in A} \mu_i - \sum_{i \in A'} \mu_i + \sum_{j \in B} \nu_j - \sum_{j \in B'} \nu_j + \sum_{k \in C} \lambda_k - \sum_{k \in C'} \lambda_k$$

where $A, A', B, B', C, C' \subseteq [n] := \{1, 2, \ldots, n\}$ satisfy

   (I) $A \cap A' = B \cap B' = C \cap C' = \emptyset$
  (II) $|A| = |B'| + |C'|, |B| = |A'| + |C'|, |C| = |A'| + |B'|$
 (III) There exists $A_1, A_2, B_1, B_2, C_1, C_2 \subseteq [n]$ such that:
     (1) $|A_1| = |A_2| = |A'|, |B_1| = |B_2| = |B'|, |C_1| = |C_2| = |C'|$
     (2) $c_{\tau(A_1),\tau(A_2)}^{\tau(A')}, c_{\tau(B_1),\tau(B_2)}^{\tau(B')}, c_{\tau(C_1),\tau(C_2)}^{\tau(C')} > 0$
     (3) $c_{\tau(B_1),\tau(C_2)}^{\tau(A)}, c_{\tau(C_1),\tau(A_2)}^{\tau(B)}, c_{\tau(A_1),\tau(B_2)}^{\tau(C)} > 0.$

This family contains a number of simpler-to-state subfamilies, including the Horn inequalities (2) and those considered in [6]; see Proposition 2.6. This is our main result:

Theorem 1.3. $(\mu, \nu, \lambda) \in \mathsf{Par}_n^3$ *satisfies (3) if* $N_{\mu,\nu,\lambda} > 0$.

We prove Theorem 1.3 in Section 2. Another necessary condition for $N_{\mu,\nu,\lambda} > 0$ is a parity requirement [6, Lemma 2.2]:

$$(4) \qquad\qquad\qquad |\mu| + |\nu| + |\lambda| \equiv 0 \ (\mathrm{mod} \ 2).$$

Let $\mathcal{G}_n$ be the tuples (A, A', B, B', C, C') satisfying (I)–(III). We believe that (3) combined with (4) provides a classification.

Conjecture 1.4. *If* $(\mu, \nu, \lambda) \in \mathsf{Par}_n^3$ *satisfies (4), and (3) holds for every* $(A, A', B, B', C, C') \in \mathcal{G}_n$, *then* $N_{\mu,\nu,\lambda} > 0$.

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[1]Exploration of this point may appear elsewhere.

This is exhaustively computer-checked, with D. Brewster's assistance, for up to $n \leq 4$ and $|\mu|, |\nu|, |\lambda| \leq 20$, for $n = 5$ and $|\mu|, |\nu|, |\lambda| \leq 16$, and for $n = 6$ and $|\mu|, |\nu|, |\lambda| \leq 12$.

Since the extended Horn inequalities are homogeneous in μ_i, ν_j, λ_k, Conjecture 1.4 immediately implies the *Newell-Littlewood saturation conjecture* [6, Conjecture 5.4]:

(5) $\qquad\qquad$ If $(\mu, \nu, \lambda) \in \mathrm{Par}_n^3$ and (4) holds, then $N_{t\mu, t\nu, t\lambda} > 0 \Rightarrow N_{\mu, \nu, \lambda} > 0$,

where $t \in \mathbb{Z}_{>0}$ and $t\mu = (t\mu_1, t\mu_2, \ldots) \in \mathrm{Par}_n$. However, unlike the situation of [10], we have no proof that (5) $\Rightarrow$ Conjecture 1.4.

The Newell-Littlewood numbers enjoy a symmetry ([6, Lemma 2.2(I)]), namely, that

(6) $\qquad\qquad N_{\mu^{(1)}, \mu^{(2)}, \mu^{(3)}} = N_{\mu^{(\sigma(1))}, \mu^{(\sigma(2))}, \mu^{(\sigma(3))}}$, for any $\sigma \in \mathfrak{S}_3$.

By construction, the extended Horn inequalities respect this $\mathfrak{S}_3$-symmetry. It is also evident from the definition that $\mathcal{G}_n \subset \mathcal{G}_{n+1}$.

In Section 3 we will prove the "Pieri case" of Conjecture 1.4.

Theorem 1.5. *Conjecture 1.4 is true when at least one of μ, ν, λ is a row or a column.*

In contrast with [2, 15], our methods are completely combinatorial, starting from (1). The main work was the uncovering of the form of the inequalities (3).

2. PROOF OF THEOREM 1.3, SUBFAMILIES, AND STABILITY

2.1. **Proof of Theorem 1.3:** We need this result of E. Briand-R. Orellana-M. Rosas:

Theorem 2.1. [4, Theorem 4] *For any partition λ, μ and ν such that $\lambda \subseteq (n^{k+l}), \mu \subseteq (n^k)$ and $\nu \subseteq (n^l)$,*

$$c_{\mu,\nu}^{\lambda} = c_{\mu^{\vee(n^k)}, \nu^{\vee(n^l)}}^{\lambda^{\vee(n^{k+l})}},$$

where if $\theta \subseteq (n^m)$, $\theta^{\vee(n^m)}$ is the partition obtained by taking the complement of $\theta \subseteq n \times m$ and rotating 180-degrees.

We use the following reformulation of the main definition.

Lemma 2.2. *In Definition 1.2, it is equivalent to replace Condition (III)(3) with*

(III)(3)′ $m_A := \min(|B'|, |C'|)$, $m_B := \min(|A'|, |C'|)$, $m_C := \min(|A'|, |B'|)$

$$0 < c_{\tau(B_1^c \cup [n+1, n+|B_1|-m_A]), \tau(C_2^c \cup [n+1, n+|C_2|-m_A])}^{\tau(A^c \cup [n+1, n+|A|-m_A])},$$

$$0 < c_{\tau(C_1^c \cup [n+1, n+|C_1|-m_B]), \tau(A_2^c \cup [n+1, n+|A_2|-m_B])}^{\tau(B^c \cup [n+1, n+|B|-m_B])},$$

$$0 < c_{\tau(A_1^c \cup [n+1, n+|A_1|-m_C]), \tau(B_2^c \cup [n+1, n+|B_2|-m_C])}^{\tau(C^c \cup [n+1, n+|C|-m_C])}.$$

$\qquad$ *(Above $A^c, B^c, C^c \subseteq [n]$.)*

Proof. Notice that

$$\tau(C^c \cup [n+1, n+|C|-m_C])$$
$$= \tau(C^c) \cup (n+1-(n-|C|+1))^{|C|-m_C}$$
$$= \tau(C^c) \cup |C|^{|C|-m_C}$$

Now, $\tau(C^c)$ is in fact $\tau(C)^\vee$ *and transposed*. Thus

$$\tau(C^c) \cup |C|^{|C|-m_C} = (\tau(C)^\vee + (|C| - m_C)^{|C|})'.$$

A similar equality holds for the other two arguments. Hence

$$c^{\tau(C^c \cup [n+1, n+|C|-m_C])}_{\tau(A_1^c \cup [n+1, n+|A_1|-m_C]), \tau(B_2^c \cup [n+1, n+|B_2|-m_C])}$$
$$= c^{(\tau(C)^\vee + |C|^{|C|-m_C})'}_{(\tau(A_1)^\vee + |A_1|^{|A_1|-m_C})', (\tau(B_2)^\vee + |B_2|^{|B_2|-m_C})'}$$
$$= c^{\tau(C)^\vee + (|C|-m_C)^{|C|}}_{\tau(A_1)^\vee + (|A_1|-m_C)^{|A_1|}, \tau(B_2)^\vee + (|B_2|-m_C)^{|B_2|}},$$

where we have used the standard symmetry $c^{\gamma}_{\alpha,\beta} = c^{\gamma'}_{\alpha',\beta'}$.

Since

$$\tau(C)^\vee \subseteq (n - |C|)^{|C|}, \tau(A_1)^\vee \subseteq (n - |A_1|)^{|A_1|} \text{ and } \tau(B_2)^\vee \subseteq (n - |B_2|)^{|B_2|},$$

one has

$$\tau(C)^\vee + (|C| - m_C)^{|C|} \subseteq (n - m_C)^{|C|},$$
$$\tau(A_1)^\vee + (|A_1| - m_C)^{|A_1|} \subseteq (n - m_C)^{|A_1|},$$
$$\tau(B_2)^\vee + (|B_2| - m_C)^{|B_2|} \subseteq (n - m_C)^{|B_2|}.$$

Observe

$$(\tau(C)^\vee + (|C| - m_C)^{|C|})^{\vee((n-m_C)^{|C|})} = \tau(C),$$
$$(\tau(A_1)^\vee + (|A_1| - m_C)^{|A_1|})^{\vee((n-m_C)^{|A_1|})} = \tau(A_1),$$
$$(\tau(B_2)^\vee + (|C| - m_C)^{|B_2|})^{\vee((n-m_C)^{|B_2|})} = \tau(B_2).$$

Since, by Condition (II), $|C| = |A_1| + |B_2|$, we can apply Theorem 2.1 and obtain

$$c^{\tau(C)^\vee + (|C|-m_C)^{|C|}}_{\tau(A_1)^\vee + (|A_1|-m_C)^{|A_1|}, \tau(B_2)^\vee + (|B_2|-m_C)^{|B_2|}} = c^{\tau(C)}_{\tau(A_1), \tau(B_2)}.$$

The other two cases are similarly proved to be equivalent with the corresponding condition in Definition 1.2. $\qquad\square$

Let $(A, A', B, B', C, C') \in \mathcal{G}_n$, and let $(A_1, A_2, B_1, B_2, C_1, C_2)$ be as in (III). Let $\mu, \nu, \lambda \in \mathrm{Par}_n$ satisfy $N_{\mu,\nu,\lambda} > 0$. By (1), there exist $\alpha, \beta, \gamma \in \mathrm{Par}_n$ such that

$$c^\mu_{\alpha,\beta}, c^\nu_{\alpha,\gamma}, c^\lambda_{\beta,\gamma} > 0.$$

By Theorem 1.1, $(\mu, \alpha, \beta), (\nu, \alpha, \gamma),$ and (λ, β, γ) satisfy (2). In particular,

$$(7) \qquad \sum_{i \in A'} \mu_i \le \sum_{i \in A_1} \beta_i + \sum_{i \in A_2} \alpha_i, \sum_{j \in B'} \nu_j \le \sum_{j \in B_1} \alpha_j + \sum_{j \in B_2} \gamma_j, \sum_{k \in C'} \lambda_k \le \sum_{k \in C_1} \gamma_k + \sum_{k \in C_2} \beta_k$$

In addition, since $\mu, \alpha, \beta \in \mathrm{Par}_n$, and in view of Lemma 2.2,

$$\sum_{i \in A} \mu_i = |\mu| - \sum_{i \in A^c} \mu_i$$
$$= |\mu| - \sum_{i \in A^c \cup [n+1, n+|A|-m_A]} \mu_i$$

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$$\geq |\mu| - \sum_{i \in B_1^c \cup [n+1, n+|B_1|-m_A]} \alpha_i - \sum_{i \in C_2^c \cup [n+1, n+|C_2|-m_A]} \beta_i$$

$$= |\mu| - \sum_{i \in B_1^c} \alpha_i - \sum_{i \in C_2^c} \beta_i$$

$$= |\alpha| - \sum_{i \in B_1^c} \alpha_i + |\beta| - \sum_{i \in C_2^c} \beta_i$$

$$= \sum_{i \in B_1} \alpha_i + \sum_{i \in C_2} \beta_i.$$

By the same logic,

$$\sum_{j \in B} \nu_j \geq \sum_{j \in C_1} \gamma_j + \sum_{j \in A_2} \alpha_j \text{ and } \sum_{k \in C} \gamma_k \geq \sum_{k \in A_1} \beta_k + \sum_{k \in B_2} \gamma_k.$$

Therefore,

$$\sum_{i \in A'} \mu_i + \sum_{j \in B'} \nu_j + \sum_{k \in C'} \lambda_k \leq \left(\sum_{i \in A_1} \beta_i + \sum_{i \in A_2} \alpha_i \right) + \left(\sum_{j \in B_1} \alpha_j + \sum_{j \in B_2} \gamma_j \right) + \left(\sum_{k \in C_1} \gamma_k + \sum_{k \in C_2} \beta_k \right)$$

$$= \left(\sum_{i \in B_1} \alpha_i + \sum_{i \in C_2} \beta_i \right) + \left(\sum_{j \in C_1} \gamma_j + \sum_{j \in A_2} \alpha_j \right) + \left(\sum_{k \in A_1} \beta_k + \sum_{k \in B_2} \gamma_k \right)$$

$$\leq \sum_{i \in A} \mu_i + \sum_{j \in B} \nu_j + \sum_{k \in C} \lambda_k.$$

$\square$

Remark 2.3. Using the above argument, one can show that other inequalities of the form (3) hold. For example, we can replace (III) by (III)(3)′ and replace (II) by

(II)′ $|A| \geq \max(|B'|, |C'|), |B| \geq \max(|A'|, |C'|), |C| \geq \max(|A'|, |B'|).$

Any $(\mu, \nu, \lambda) \in \mathsf{Par}_n^3$ such that $N_{\mu,\nu,\lambda} > 0$ satisfies the corresponding inequality.

2.2. **Special subclasses of the inequalities.** In [6] we proved:

Theorem 2.4 (Extended Weyl inequalities). *Let* $(\mu, \nu, \lambda) \in \mathsf{Par}_n^3$ *and* $1 \leq k \leq i < j \leq l \leq n$, *let* $m = \min(i - k, l - j)$ *and* $M = \max(i - k, l - j)$. *If* $N_{\mu,\nu,\lambda} > 0$ *then*

(8) $$\mu_i - \mu_j \leq \lambda_k - \lambda_l + \nu_{m-p+1} + \nu_{M+p+2}, \quad \text{where } 0 \leq p \leq m.$$

Definition 2.5. *For disjoint* $X, Y \subseteq [n]$, *the subset-sum inequalities are*

(9) $$0 \leq \sum_{i \in X} \mu_i + \sum_{i \in Y} \nu_i - \sum_{i \in Y} \mu_i - \sum_{i \in X} \nu_i + \sum_{i=1}^{|X|+|Y|} \lambda_i.$$

Proposition 2.6. *Any* $(\mu, \nu, \lambda) \in \mathsf{Par}_n^3$ *that satisfy all of the extended Horn inequalities* (3) *also satisfy all* $\mathfrak{S}_3$-*permutations* (6) *of:*

  (i) *the Horn inequalities* (2);
 (ii) *the extended Weyl inequalities* (8);
(iii) *the subset-sum inequalities* (9);
 (iv) *the triangle inequalities on* $|\mu|, |\nu|, \text{and } |\lambda|$;
  (v) $|\mu \wedge \nu| \geq \frac{|\mu|+|\nu|-|\lambda|}{2}$.

Proof. A Horn inequality is of the form

$$\sum_{i \in X} \mu_i \leq \sum_{j \in Y} \nu_j + \sum_{k \in Z} \lambda_k$$

for $X, Y, Z \subseteq [n]$ and $c^{\tau(X)}_{\tau(Y), \tau(Z)} > 0$. Letting

$$(A, A', B, B', C, C') := (\emptyset, X, Y, \emptyset, Z, \emptyset)$$

with $(A_1, A_2, B_1, B_2, C_1, C_2)$ as $(Z, Y, \emptyset, \emptyset, \emptyset, \emptyset)$ shows this is an extended Horn inequality. The Littlewood-Richardson positivity conditions (III)(2) and (III)(3) clearly hold. Any $\mathfrak{S}_3$-symmetry of the Horn inequality is of the form (3), since being of the form (3) is evidently preserved under $\mathfrak{S}_3$.

Similarly, let

$$(A, A', B, B', C, C') := (\{j\}, \{i\}, \{m - p + 1, M + p + 2\}, \emptyset, \{k\}, \{l\})$$

with $(A_1, A_2, B_1, B_2, C_1, C_2)$ as $(\{k\}, \{i - k + 1\}, \emptyset, \emptyset, \{l - j + 1\}, \{j\})$. Let us only comment on the assertion $c^{\tau(B)}_{\tau(C_1), \tau(A_2)} = c^{(M+p, m-p)}_{(l-j), (i-k)} > 0$, which is true by Pieri's rule. Thus, (8) is of the form (3).

The subset-sum inequalities are of type (3). Let

$$(A, A', B, B', C, C') := (X, Y, Y, X, [|X| + |Y|], \emptyset)$$

with $(A_1, A_2, B_1, B_2, C_1, C_2) := ([|Y|], Y, X, [|X|], \emptyset, \emptyset)$. By letting $X = [n]$ and $Y = \emptyset$, the triangle inequalities are cases of the subset-sum inequalities. The verification is clear.

Let $X := \{i \in [n] : \mu_i \leq \nu_i\}$, and let $Y := \{i \in [n] : \mu_i > \nu_i\} = X^c$. Then

$$|\mu \wedge \nu| = \sum_{i \in X} \mu_i + \sum_{j \in Y} \nu_j,$$

and so (v) can be rewritten as

$$2 \sum_{i \in X} \mu_i + 2 \sum_{j \in Y} \nu_j \geq \sum_{i=1}^{n} \mu_i + \sum_{j=1}^{n} \nu_j - \sum_{k=1}^{n} \lambda_k$$

or

$$0 \leq \sum_{i \in X} \mu_i - \sum_{i \in Y} \mu_i + \sum_{j \in Y} \nu_j - \sum_{j \in X} \nu_j + \sum_{k=1}^{n} \lambda_k$$

This is a subset-sum inequality, and we are done by (iii). □

Example 2.7. The extended Horn inequalities for $n = 2$ are the $\mathfrak{S}_3$-permutations (6) of:

(10) $$0 \leq \mu_1 + \nu_1 - \lambda_1$$

(11) $$0 \leq \mu_1 + \nu_2 - \lambda_2$$

(12) $$0 \leq \mu_1 + \mu_2 + \nu_1 + \nu_2 - \lambda_1 - \lambda_2$$

(13) $$0 \leq \mu_1 - \mu_2 - \nu_1 + \nu_2 + \lambda_1 + \lambda_2$$

where (10) and (11) are Horn inequalities, (12) is a triangle inequality, and (13) is both an extended Weyl inequality and a subset-sum inequality.

Example 2.8. The extended Horn inequalities for $n = 3$ are the $\mathfrak{S}_3$-permutations of the $n = 3$ Horn inequalities, extended Weyl inequalities, subset-sum inequalities,

$$(14) \qquad 0 \leq -\mu_1 + \mu_2 + \mu_3 + \nu_1 - \nu_2 + \nu_3 + \lambda_1 + \lambda_2 - \lambda_3,$$

$$(15) \qquad 0 \leq \mu_1 - \mu_2 + \mu_3 + \nu_1 - \nu_2 + \nu_3 + \lambda_1 - \lambda_2 + \lambda_3,$$

$$(16) \qquad 0 \leq \mu_1 - \mu_2 + \nu_1 - \nu_2 + \lambda_2 + \lambda_3,$$

$$(17) \qquad 0 \leq \mu_2 - \mu_3 + \nu_2 - \nu_3 + \lambda_2 + \lambda_3.$$

Example 2.9 (Minimal inequalities?). The Horn inequalities (2) are redundant. One can shorten the list to those where

$$c_{\tau(I),\tau(J)}^{\tau(K)} = 1;$$

this is a result of P. Belkale [1, Theorem 9] (conjectured by C. Woodward). That each of these inequalities are essential is proved by A. Knutson-T. Tao-C. Woodward [11, Section 6]. We know of no naïve analogue of these results. Specifically, the inequalities (15), (16), and (17) are redundant, since they are implied by the "partition inequalities", *i.e.*, $\mu, \nu, \lambda \in \mathsf{Par}_3$. However, all Littlewood-Richardson coefficients associated with (16) and (17) are 1.

Proposition 2.10. *Conjecture 1.4 holds for $n = 2$.*

Proof. We prove the contrapositive. Suppose $N_{\mu,\nu,\lambda} = 0$ and (4) holds. By [6, Theorem 5.14] either an $n = 2$ Horn inequality or extended Weyl inequality fails. By Proposition 2.6(ii), the inequalities (3) include the Horn inequalities and the extended Weyl inequalities. Therefore, an inequality (3) is violated. $\qquad \square$

3. PROOF OF THEOREM 1.5

First consider the case where one of the partitions is a single row. Without loss of generality, $\lambda = (p)$. By Proposition 2.6(i), (μ, ν, λ) satisfies all $\mathfrak{S}_3$ permutations of the Horn inequalities. In particular, they satisfy $\mu_{i+1} \leq \nu_i + \lambda_2$ and $\nu_{i+1} \leq \mu_i + \lambda_2$ for all $i \in [n-1]$. This implies that $\mu_{i+1} \leq \nu_i$ and $\nu_{i+1} \leq \mu_i$, so $\mu_{i+1}, \nu_{i+1} \leq (\mu \wedge \nu)_i$ for all $i \in [n-1]$, which is equivalent to saying that $\mu/(\mu \wedge \nu)$ and $\nu/(\mu \wedge \nu)$ are horizontal strips.

Let $k := \frac{|\mu|+|\nu|-p}{2}$. Since Proposition 2.6(iv) says that (μ, ν, λ) satisfies the triangle inequalities, $k \geq 0$. Moreover, $|\mu| + |\nu| + |\lambda|$ is even (by hypothesis), hence $k \in \mathbb{Z}_{\geq 0}$. Proposition 2.6(v) also says that $k \leq |\mu \wedge \nu|$.

Claim 3.1. *There exist at least $|\mu \wedge \nu| - k$ columns i such that $\mu_i' = \nu_i' > 0$.*

Proof. Without loss of generality, say that $\mu_1 \leq \nu_1$. Since $\mu/(\mu \wedge \nu)$ is a horizontal strip, there are $|\mu/(\mu \wedge \nu)| = |\mu| - |\mu \wedge \nu|$ columns i such that $\mu_i' > \nu_i'$, and similarly there are $|\nu| - |\mu \wedge \nu|$ columns i such that $\mu_i' < \nu_i'$. Since there are ν_1 columns where at least one of μ_i' or ν_i' is nonzero, this means that there are $\nu_1 - |\mu| - |\nu| + 2|\mu \wedge \nu|$ columns such that $\mu_i' = \nu_i' > 0$, so it suffices to prove that

$$\nu_1 - |\mu| - |\nu| + 2|\mu \wedge \nu| \geq |\mu \wedge \nu| - k.$$

Rearranging the terms and substituting in for the definition of k, this becomes

$$(18) \qquad 0 \leq \nu_1 - |\mu| - |\nu| + |\mu \wedge \nu| + \frac{|\mu| + |\nu| - p}{2}.$$

Define
$$X := \{i \in [n] : \mu_i > \nu_i\} \text{ and } Y := \{i \in [n] : \mu_i \leq \nu_i\} = X^c.$$
By assumption, $1 \in Y$. Hence from (18) and the definition of $|\cdot|$,
$$0 \leq 2\nu_1 - 2|\mu| - 2|\nu| + 2|\mu \wedge \nu| + (|\mu| + |\nu| - p)$$
$$= 2\nu_1 - |\mu| - |\nu| + 2|\mu \wedge \nu| - p$$

$$= 2\nu_1 - \sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \nu_i + 2\sum_{i \in X} \nu_i + 2\sum_{i \in Y} \mu_i - p$$
$$= 2\nu_1 - \sum_{i \in X} \mu_i - \sum_{i \in Y} \nu_i + \sum_{i \in X} \nu_i + \sum_{i \in Y} \mu_i - p$$
$$= -\sum_{i \in X} \mu_i - \sum_{i \in Y \setminus \{1\}} \nu_i + \sum_{i \in X \cup \{1\}} \nu_i + \sum_{i \in Y} \mu_i - p.$$
However, this is always true, since
$$(Y, X, X \cup \{1\}, Y \setminus \{1\}, [n] \setminus \{1\}, \{1\}) \in \mathcal{G}_n,$$
which can be seen by letting
$$(A_1, A_2, B_1, B_2, C_1, C_2) =$$
$$([|X| + 1] \setminus \{1\}, \{i - 1 : i \in X\}, \{i - 1 \in \mathbb{Z}_{>0} : i \in Y\}, [|Y|] \setminus \{1\}, \{1\}, \{1\}).$$
The verification of $c_{\tau(C_1), \tau(A_2)}^{\tau(B)} > 0$ relies on
$$\tau(\{i - 1 : i \in X\}) = \tau(X \cup \{1\}).$$
Similarly one checks $c_{\tau(B_1), \tau(C_2)}^{\tau(A)} > 0$. $\qquad\qquad\square$

Let α be the partition formed by removing the southernmost box from the $|\mu \wedge \nu| - k$ rightmost columns i of $\mu \wedge \nu$ such that $\mu_i' = \nu_i' > 0$. Since $\mu \wedge \nu = (\mu' \wedge \nu')'$, the boxes removed from $\mu \wedge \nu$ to form α are in different columns than the boxes removed from μ to form $\mu \wedge \nu$ or from ν to form $\mu \wedge \nu$. Thus, μ/α and ν/α are both horizontal strips. Also,
$$|\mu/\alpha| = |\mu/(\mu \wedge \nu)| + |(\mu \wedge \nu)/\alpha|$$
$$= (|\mu| - |\mu \wedge \nu|) + (|\mu \wedge \nu| - k)$$
$$= |\mu| - \frac{|\mu| + |\nu| - p}{2}$$
$$= \frac{|\mu| + p - |\nu|}{2}$$
and similarly $|\nu/\alpha| = \frac{|\nu| + p - |\mu|}{2}$.

As a result, one can remove a horizontal strip from μ of length $\frac{|\mu| + p - |\nu|}{2}$, and then add a horizontal strip of length $\frac{|\nu| + p - |\mu|}{2}$ back in to result in ν. This is exactly the statement of Proposition 2.4 of [6], so $N_{\mu, \nu, (p)} > 0$.

Now consider the case where one of the partitions is a single column. Without loss of generality, $\lambda = (1^p)$. By Proposition 2.6, (μ, ν, λ) satisfies all $\mathfrak{S}_3$ permutations of the Horn inequalities. In particular, they satisfy $\mu_i \leq \nu_i + \lambda_1$ and $\nu_i \leq \mu_i + \lambda_1$ for all $i \in [n]$. This

implies that $\mu_i \leq \nu_i + 1$ and $\nu_i \leq \mu_i + 1$, so $\mu_i, \nu_i \leq (\mu \wedge \nu)_i + 1$ for all $i \in [n+1]$, which is equivalent to saying that $\mu/(\mu \wedge \nu)$ and $\nu/(\mu \wedge \nu)$ are vertical strips.

Let $k := \frac{|\mu| + |\nu| - p}{2}$. As before, $k \in \mathbb{Z}_{\geq 0}$ and $k \leq |\mu \wedge \nu|$.

Claim 3.2. *There exist at least $|\mu \wedge \nu| - k$ rows i such that $\mu_i = \nu_i > 0$.*

Proof. Since $\mu/(\mu \wedge \nu)$ is a vertical strip, there are $|\mu/(\mu \wedge \nu)| = |\mu| - |\mu \wedge \nu|$ rows i such that $\mu_i > \nu_i$, and similarly there are $|\nu| - |\mu \wedge \nu|$ rows i such that $\mu_i < \nu_i$. Let $L = \max(\ell(\mu), \ell(\nu))$. Since there are L rows where at least one of μ_i or ν_i is nonzero, this means that there are $L - |\mu| - |\nu| + 2|\mu \wedge \nu|$ rows such that $\mu_i = \nu_i > 0$, so it suffices to prove that

$$L - |\mu| - |\nu| + 2|\mu \wedge \nu| \geq |\mu \wedge \nu| - k.$$

Rearranging the terms and substituting in for the definition of k, this becomes

$$0 \leq L - |\mu| - |\nu| + |\mu \wedge \nu| + \frac{|\mu| + |\nu| - p}{2}.$$

Define
$$X := \{i \in [L] : \mu_i > \nu_i\} \text{ and } Y := \{i \in [L] : \mu_i \leq \nu_i\} = [L] \setminus X.$$
Multiplying the above expression by 2 and using the definition of $|\cdot|$, we get:

$$0 \leq 2L - 2|\mu| - 2|\nu| + 2|\mu \wedge \nu| + (|\mu| + |\nu| - p)$$
$$= 2L - |\mu| - |\nu| + 2|\mu \wedge \nu| - p$$
$$= 2L - \sum_{i=1}^{L} \mu_i - \sum_{i=1}^{L} \nu_i + 2\sum_{i \in X} \nu_i + 2\sum_{i \in Y} \mu_i - p$$
$$= 2L - \sum_{i \in X} \mu_i - \sum_{i \in Y} \nu_i + \sum_{i \in X} \nu_i + \sum_{i \in Y} \mu_i - p$$

We split the remainder of the proof of the claim into two cases: whether $L < p$ or $L \geq p$.

Case 1: ($L < p$) Here we can rewrite the above inequality as

$$0 \leq -\sum_{i \in X} \mu_i - \sum_{i \in Y} \nu_i + \sum_{i \in X \cup ([p] \setminus [L])} \nu_i + \sum_{i \in Y \cup ([p] \setminus [L])} \mu_i + L - (p - L)$$

However, this is always true, since

$$(Y \cup ([p] \setminus [L])), X, X \cup ([p] \setminus [L]), Y, [L], [p] \setminus [L]) \in \mathcal{G}_n$$

which can be seen by letting

$$(A_1, A_2, B_1, B_2, C_1, C_2) = ([|X|], X, Y, [|Y|], [|Y| + p - L] \setminus [|Y|], [|X| + p - L] \setminus [|X|]).$$

The slightly trickier verification needed from (III)(3) is

$$c_{\tau(C_1), \tau(A_2)}^{\tau(B)} = c_{\tau([|Y| - p + L] - [|Y|]), \tau(X)}^{\tau(X \cup ([p] - [L]))} = c_{|Y|^{p-L}, \tau(X)}^{\tau(X) + (L - |X|)^{p-L}} = c_{|Y|^{p-L}, \tau(X)}^{\tau(X) + |Y|^{p-L}} > 0;$$

the latter is obvious. The check $c_{\tau(B_1), \tau(C_2)}^{\tau(A)} > 0$ is analogous.

Case 2: ($L \geq p$) Here we can instead rewrite the above inequality as

$$0 \leq 2(L - p) - \sum_{i \in X} \mu_i - \sum_{i \in Y} \nu_i + \sum_{i \in X} \nu_i + \sum_{i \in Y} \mu_i + p,$$

and so it suffices to show

$$0 \leq -\sum_{i \in X} \mu_i - \sum_{i \in Y} \nu_i + \sum_{i \in X} \nu_i + \sum_{i \in Y} \mu_i + p.$$

This is true since $(Y, X, X, Y, [L], \emptyset) \in \mathcal{G}_n$ is just a subset-sum inequality. $\square$

Let α be the partition formed by removing the rightmost box from the $|\mu \wedge \nu| - k$ southernmost rows i of $\mu \wedge \nu$ such that $\mu_i = \nu_i > 0$. Since the boxes removed from $\mu \wedge \nu$ to form α are in different rows than the boxes removed from μ to form $\mu \wedge \nu$ or from ν to form $\mu \wedge \nu$, μ/α and ν/α are both vertical strips. In addition,

$$\begin{aligned}
|\mu/\alpha| &= |\mu/(\mu \wedge \nu)| + |(\mu \wedge \nu)/\alpha| \\
&= (|\mu| - |\mu \wedge \nu|) + (|\mu \wedge \nu| - k) \\
&= |\mu| - \frac{|\mu| + |\nu| - p}{2} \\
&= \frac{|\mu| + p - |\nu|}{2}
\end{aligned}$$

and similarly $|\nu/\alpha| = \frac{|\nu| + p - |\mu|}{2}$.

As a result, one can remove a vertical strip from μ of length $\frac{|\mu| + p - |\nu|}{2}$, and then add a vertical strip of length $\frac{|\nu| + p - |\mu|}{2}$ back in to result in ν. This is exactly the conjugate statement of Proposition 2.4 of [6], so $N_{\mu,\nu,(1^p)} > 0$. $\square$

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