# Computational complexity, Newton polytopes, and Schubert polynomials

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**Abstract.** The nonvanishing problem asks if a coefficient of a polynomial is nonzero. Many families of polynomials in algebraic combinatorics admit combinatorial counting rules and simultaneously enjoy having *saturated Newton polytopes* (SNP). Thereby, in amenable cases, nonvanishing is in the complexity class $\mathsf{NP} \cap \mathsf{coNP}$ of problems with "good characterizations". This suggests a new algebraic combinatorics viewpoint on complexity theory.

This paper focuses on the case of *Schubert polynomials*. These form a basis of all polynomials and appear in the study of cohomology rings of flag manifolds. We give a tableau criterion for nonvanishing, from which we deduce the first polynomial time algorithm. These results are obtained from new characterizations of the *Schubitope*, a generalization of the permutahedron defined for any subset of the $n \times n$ grid, together with a theorem of A. Fink, K. Mészáros, and A. St. Dizier (2018), which proved a conjecture of C. Monical, N. Tokcan, and the third author (2017).

**Keywords:** Schubert polynomials, Newton polytopes, computational complexity

# 1 Introduction

The main results of this extended abstract of [1, 2] concern Schubert polynomials; these are found in Section 2. Those results illustrate a general algebraic combinatorics paradigm for computational complexity theory that we wish to put forward here.

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## 1.1 Nonvanishing decision problems and SNP

Algebraic combinatorics studies families of polynomials parameterized by combinatorial objects $\diamondsuit$

$$F_{\diamondsuit} = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_{\alpha,\diamondsuit} x^{\alpha} = \sum_{s \in \mathcal{S}} \mathsf{wt}(s) \in \mathbb{Z}[x_1, x_2, \ldots, x_n],$$

each viewed as the multivariate weight generating series for some combinatorially defined set $\mathcal{S}$.

**Example 1.1** (Schur polynomials)**.** *$F_{\diamondsuit} = s_{\lambda}$ is a* Schur polynomial, *where $\diamondsuit = \lambda$ is an integer partition. Here, $\mathcal{S}$ is the set of* semistandard Young tableaux *of shape $\lambda$ with entries in $[n]$, and $\mathsf{wt}(s) = \prod_i x_i^{\# i \in s}$. Schur polynomials are an important basis of the vector space of all symmetric polynomials.*

**Example 1.2** (Stanley's chromatic symmetric polynomial)**.** *Another symmetric polynomial is Stanley's* chromatic polynomial *$F_{\diamondsuit} = \chi_G$ [20]. This time $\diamondsuit = G = (V, E)$ is a simple graph, $\mathcal{S}$ is the set of proper $n$-colorings of $G$, i.e., functions $s : V \to \{1, 2, \ldots, n\}$ such that $s(i) \neq s(j)$ if $\{i, j\} \in E$, and $\mathsf{wt}(s) = \prod_i x_i^{\# s^{-1}(i)}$.*

**Example 1.3** (Schubert polynomials)**.** *The central example of this paper is non-symmetric. It is the family of* Schubert polynomials *$F_{\diamondsuit} = \mathfrak{S}_w$, a basis of all polynomials. Now, $\diamondsuit = w$ is a permutation. There are many choices for $\mathcal{S}$, such as the* reduced compatible sequences *of [5]. Definitions are given in Section 2.*

**Problem 1.4** (nonvanishing)**.** *What is the complexity of deciding $c_{\alpha,\diamondsuit} \neq 0$, as measured in the input size of $\alpha$ and $\diamondsuit$ (under the assumption that arithmetic operations take constant time)?*

In this paper, we give a polynomial time algorithm to determine $c_{\alpha,w} \neq 0$ for the Schubert polynomial. In general, nonvanishing may be undecidable: fix $S \subseteq \mathbb{N}$ that is not recursively enumerable, and let $F_m = \sum_{i=1}^{m} c_{i,m} x^m$ with $c_{i,m} = 1$ if $i \in S$ and 0 otherwise. Such sets $S$ exist because there are uncountably many subsets of $\mathbb{N}$, but only countably many algorithms. One can explicitly take $S$ to be the set of halting Turing machines under some numerical encoding [21], or the set of Gödel encodings [11] of statements about $(\mathbb{N}, +, \times)$ provable in first-order Peano arithmetic. All this said, in our cases of interest, $c_{\alpha,\diamondsuit} \in \mathbb{Z}_{\geq 0}$ has *combinatorial positivity*: it is given by a counting rule that implies nonvanishing is in the class NP of problems with a polynomial time checkable certificate of a YES decision.

Evidently, nonvanishing concerns the *Newton polytope*,

$$\mathsf{Newton}(F_{\diamondsuit}) = \mathrm{conv}\{\alpha : c_{\alpha,\diamondsuit} \neq 0\} \subseteq \mathbb{R}^n.$$

C. Monical, N. Tokcan and the third author [17] showed that for many examples, $F_{\diamondsuit}$ has *saturated Newton polytope* (SNP), i.e., $\gamma \in \mathsf{Newton}(F_{\diamondsuit}) \cap \mathbb{Z}^n \iff c_{\gamma,\diamondsuit} \neq 0$. The relevance of SNP to Problem 1.4 is:

> SNP $\Rightarrow$ $\mathsf{nonvanishing}(F_\diamondsuit)$ is equivalent to checking membership of a lattice point in $\mathsf{Newton}(F_\diamondsuit)$.

**Example 1.5** ($\mathsf{nonvanishing}(s_\lambda)$ is in $\mathsf{P}$)**.** *$\mathsf{Newton}(s_\lambda)$ is the $\lambda$-*permutahedron *$\mathcal{P}_\lambda$, the convex hull of the $S_n$-orbit of $\lambda \in \mathbb{R}^n$. By symmetry one may assume $\alpha$ is a partition. Thus $c_{\alpha,\lambda}$ is the* Kostka coefficient*, and $c_{\alpha,\lambda} = 0$ if and only if $\alpha \leq \lambda$ in* dominance order*. So $\mathsf{nonvanishing}(s_\lambda)$ is in the class $\mathsf{P}$ of polynomial time problems.*

Does the "niceness" of combinatorial positivity and SNP transfer to complexity?

**Question 1.6.** *Under what conditions does combinatorial positivity and SNP of $\{F_\diamondsuit\}$ imply $\mathsf{nonvanishing}(F_\diamondsuit) \in \mathsf{P}$, or at least that $\mathsf{nonvanishing}(F_\diamondsuit) \notin \mathsf{NP}$-complete?*

On the other hand, $\chi_G$ is not generally SNP [17] and $\mathsf{nonvanishing}(\chi_G)$ is hard:

**Example 1.7** ($\chi_G$-nonvanishing is $\mathsf{NP}$-complete)**.** *For $\chi_G$, nonvanishing is clearly in $\mathsf{NP}$. In fact, for each fixed $n \geq 3$ it is $\mathsf{NP}$-complete. The $n$-coloring problem of deciding if a graph has an $n$-proper coloring is $\mathsf{NP}$-complete for each fixed $n \geq 3$. Given an efficient oracle to solve $\mathsf{nonvanishing}(\chi_G)$, call it on each partition of $|V|$ with $n$ parts to decide if there exists a proper $n$-coloring. This requires only $O(|V|^n)$ calls, so it is a polynomial reduction of $n$-coloring to $\mathsf{nonvanishing}(\chi_G)$.*

## 1.2 Context from computer science; connection to Stanley's Schur positivity conjecture

Examples 1.5 and 1.7 show that $\mathsf{nonvanishing}$ can achieve the extremes of $\mathsf{NP}$. What about the non-extremes?

The class $\mathsf{NP}$-intermediate consists of $\mathsf{NP}$ problems that are neither in $\mathsf{P}$ nor $\mathsf{NP}$-complete. *Ladner's theorem* states that if $\mathsf{P} \neq \mathsf{NP}$ there exists an (artificial) $\mathsf{NP}$-intermediate problem. Many natural problems from algebra, number theory, game theory and combinatorics are *suspected* to be $\mathsf{NP}$-intermediate. An example is the *Graph Isomorphism problem*.

The class $\mathsf{coNP}$ consists of problems whose complements are in $\mathsf{NP}$, i.e., those with a polynomial time checkable certificate of a `NO` decision.

> SNP $\Rightarrow$ given a halfspace description of the Newton polytope, an inequality violation checkable in polynomial time gives a $\mathsf{coNP}$ certificate.

The above implication of SNP says that any solution $\{F_\diamondsuit\}$ to the following problem gives $\mathsf{nonvanishing}(F_\diamondsuit) \in \mathsf{NP} \cap \mathsf{coNP}$.

**Problem 1.8.** *For a combinatorially positive family of SNP polynomials $\{F_\diamondsuit\}$, determine half space descriptions of $\mathsf{Newton}(F_\diamondsuit)$.*

The class $\mathsf{NP} \cap \mathsf{coNP}$ is intriguing. Membership of a problem in $\mathsf{NP} \cap \mathsf{coNP}$ sometimes foreshadows the harder proof that it is in $\mathsf{P}$. For example, consider

$$\mathsf{primes} = \text{"is a positive integer } n \text{ prime?"}$$

Clearly, $\mathsf{primes} \in \mathsf{coNP}$. In 1975, V. Pratt [18] showed $\mathsf{primes} \in \mathsf{NP}$. It was about thirty years before the celebrated discovery of the *AKS primality test* of M. Agrawal, N. Kayal, and N. Saxena [3], establishing $\mathsf{primes} \in \mathsf{P}$.

In retrospect, another example is the *linear programming* problem

$$\mathsf{LPfeasibility} = \text{"is } A\mathbf{x} = b, \mathbf{x} \geq 0 \text{ feasible?"}$$

Clearly $\mathsf{LPfeasibility} \in \mathsf{NP}$. Membership in $\mathsf{coNP}$ is a consequence of *Farkas' Lemma* (1902), which is a foundation for LP duality, conjectured by J. von Neumann and proved by G. Dantzig in 1948 (cf. [7]). Yet, it was not until 1979, with L. Khachiyan's work on the *ellipsoid method* that $\mathsf{LPfeasibility} \in \mathsf{P}$ was proved; see, e.g., the textbook [19].

These examples suggest $\mathsf{P} = \mathsf{NP} \cap \mathsf{coNP}$. However, one has *integer factorization*

$$\mathsf{factorization} = \text{"given } 1 < a < b \text{ does there exist a divisor } d \text{ of } b \text{ where } 1 \leq d \leq a\text{?"}$$

An $\mathsf{NP}$ certificate is a divisor. A $\mathsf{coNP}$ certificate is a prime factorization (verified using the AKS test). Most public key cryptography (such as RSA) relies on $\mathsf{P} \neq \mathsf{NP} \cap \mathsf{coNP}$.

The debate $\mathsf{P} \stackrel{?}{=} \mathsf{NP} \cap \mathsf{coNP}$ may be rephrased as "are problems with good characterizations in $\mathsf{P}$?". One wants new examples of members of $\mathsf{NP} \cap \mathsf{coNP}$ that are not known to be in $\mathsf{P}$. If such examples are proved to be in $\mathsf{P}$, this adds evidence for "$=$". Yet, relatively few examples are known. In addition to integer factorization, one has (decision) *Discrete Log*, *Stochastic Games* [6], *Parity Games* [14] and *Lattice Problems* [4]. (It is open whether *Graph Isomorphism* is in $\mathsf{coNP}$.) We now connect this discussion with Example 1.7.

**Problem 1.9.** *Does restricting to a subclass of graphs $G$ where $\chi_G$ is SNP (or Schur positive) change the complexity of $n$-coloring?*

**Conjecture 1.10** (R. P. Stanley [20])**.** *If $G$ is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then $\chi_G$ is Schur positive.*

**Conjecture 1.11** (C. Monical [16])**.** *If $\chi_G$ is Schur positive, then it is SNP.*

Combining these two conjectures gives

**Conjecture 1.12.** *If $G$ is claw-free then $\chi_G$ is SNP.*

If $\mathsf{coNP}$ contains an $\mathsf{NP}$-complete problem then $\mathsf{NP} = \mathsf{coNP}$ [12], solving an open problem with "$=$".[1] Now, by [13], $n$-coloring claw-free graphs is $\mathsf{NP}$-complete. Therefore:

[1] In this circumstance, the (complexity) polynomial hierarchy unexpectedly collapses to the first level.

If Conjecture 1.12 holds, Problem 1.9 and Question 1.6 are answered negatively. Moreover, a solution to Problem 1.8 proves $\textsf{nonvanishing}(\chi_{\text{claw-free } G})$ is $\textsf{coNP}$, and hence $\textsf{NP} = \textsf{coNP}$.

This suggests a new complexity-theoretic rationale for the study of $\chi_G$.

### 1.3 An algebraic combinatorics paradigm for complexity

Summarizing, we are motivated by complexity to study $\textsf{nonvanishing}$ in algebraic combinatorics. Many polynomial families $\{F_\diamondsuit\}$ have combinatorial positivity and (conjecturally) SNP [17]. Together, with a solution to Problem 1.8, $\textsf{nonvanishing} \in \textsf{NP} \cap \textsf{coNP}$.

For each family $\{F_\diamondsuit\}$, one arrives at one of four logical outcomes, depending on the complexity of $\textsf{nonvanishing}(F_\diamondsuit)$ within $\textsf{NP} \cap \textsf{coNP}$:

(I) Unknown: it is a problem, in and of itself, to find additional problems that are in $\textsf{NP} \cap \textsf{coNP}$ that are not *known* to be in $\textsf{P}$.

(II) $\textsf{P}$: Give an algorithm. It will likely illuminate some special structure, of independent combinatorial interest.

(III) $\textsf{NP}$-complete: proof solves $\textsf{NP} \stackrel{?}{=} \textsf{coNP}$ with (a suprising) "$=$".

(IV) $\textsf{NP}$-intermediate: proof solves $\textsf{NP}\text{-intermediate} \stackrel{?}{=} \emptyset$ with "$\neq$" (hence $\textsf{P} \neq \textsf{NP}$).

Our main results in Section 2 illustrate (II) for Schubert polynomials.

## 2 Main results: Schubert polynomials

*Schubert polynomials* form a linear basis of all polynomials $\mathbb{Z}[x_1, x_2, x_3, \ldots]$. They were introduced by A. Lascoux–M.-P. Schützenberger [15] to study the cohomology ring of the flag manifold. These polynomials represent the Schubert classes under the Borel isomorphism. A reference is the textbook [10].

The Schubert polynomial $\mathfrak{S}_w(x_1, \ldots, x_n)$ is defined recursively for any permutation $w \in S_n$ as follows. If $w_0 = n\ n-1\ \cdots 2\ 1$ is the longest length permutation in $S_n$, then

$$\mathfrak{S}_{w_0}(x_1, \ldots, x_n) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Otherwise, $w \neq w_0$ and there exists $i$ such that $w(i) < w(i+1)$. Then one sets

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \partial_i \mathfrak{S}_{ws_i}(x_1, \ldots, x_n),$$

where $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$, and $s_i$ is the transposition swapping $i$ and $i+1$. Since $\partial_i$ satisfies

$$\partial_i \partial_j = \partial_j \partial_i \text{ for } |i-j| > 1, \text{ and } \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1},$$

the above description of $\mathfrak{S}_w$ is well-defined. In addition, under the inclusion $\iota : S_n \hookrightarrow S_{n+1}$ defined by $w(1)\cdots w(n) \mapsto w(1)\ \cdots w(n)\ n+1$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. Thus one unambiguously refers to $\mathfrak{S}_w$ for each $w \in S_\infty = \bigcup_{n\geq 1} S_n$.

To each $w \in S_\infty$ there is a unique *code*, $\mathsf{code}(w) = (c_1, c_2, \ldots, c_L) \in \mathbb{Z}^L_{\geq 0}$, where $c_i$ counts the number of boxes in the $i$-th row of the Rothe diagram $D(w)$ of $w$. If $w$ is the identity then $\mathsf{code}(w) = \emptyset$; otherwise, $c_L > 0$ (i.e., we truncate any trailing zeroes).

Now, $c_{\alpha,w} = 0$ unless $\alpha_i = 0$ for $i > L$, and moreover, $c_{\alpha,w} \in \mathbb{Z}_{\geq 0}$. Let Schubert be the nonvanishing problem for Schubert polynomials. The `INPUT` is $\mathsf{code} = (c_1, \ldots c_L) \in \mathbb{Z}^L_{\geq 0}$ with $c_L > 0$ and $\alpha \in \mathbb{Z}^L_{\geq 0}$. Schubert returns `YES` if $c_{\alpha,w} > 0$ and `NO` otherwise.

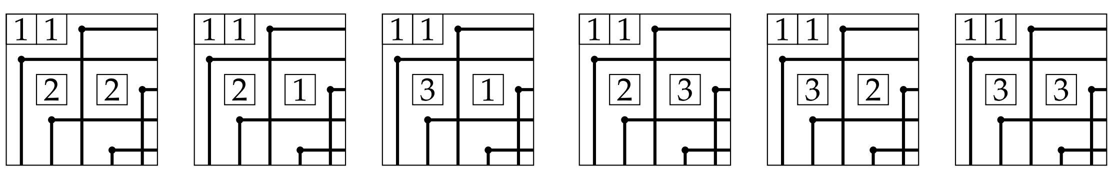
**Theorem 2.1.** Schubert $\in$ P.

We prove Theorem 2.1 using another result. For $w \in S_n$, let $\mathsf{PerfectTab}(D(w), \alpha)$ be the fillings of $D(w)$ with $\alpha_k$ many $k$'s, where entries in each column are distinct, and any entry in row $i$ is $\leq i$. Let $\mathsf{PerfectTab}_<(D(w), \alpha) \subseteq \mathsf{PerfectTab}(D(w), \alpha)$ be fillings where entries in each column increase from top to bottom.

**Theorem 2.2.** $c_{\alpha,w} > 0 \iff \mathsf{PerfectTab}(D(w), \alpha) \neq \emptyset \iff \mathsf{PerfectTab}_<(D(w), \alpha) \neq \emptyset$

In general $\#\mathsf{PerfectTab}(D(w), \alpha) \neq c_{\alpha,w}$ but rather $\#\mathsf{PerfectTab}(D(w), \alpha) \geq c_{\alpha,w}$ (cf. [9]).

**Example 2.3.** *Here are the tableaux in* $\bigcup_\alpha \mathsf{PerfectTab}_<(D(31524), \alpha)$*:*



*Hence, for instance,* $c_{(2,1,1),31524} > 0$ *but* $c_{(4),31524} = 0$.

To prove Theorems 2.1 and 2.2 we establish more general results about the *Schubitope* introduced in [17]. This polytope $\mathcal{S}_D$ generalizes the $\lambda$-permutahedron of Example 1.5. It is defined with a halfspace description for any diagram of boxes $D \subseteq [n]^2$.

In the case of Rothe diagrams $D := D(w)$, it was conjectured in [17] that $\mathcal{S}_{D(w)}$ is the Newton polytope of $\mathfrak{S}_w$ and moreover that $\mathfrak{S}_w$ has the SNP property. These conjectures were proved by A. Fink-K. Mészáros-A. St. Dizier [8]. This, combined with Theorem 3.5 and properties of perfect tableaux, proves Theorem 2.2.

*Key polynomials* $\kappa_\beta$ are a specialization of the non-symmetric Macdonald polynomials. Similarly to the above case, for skyline diagrams $D := D_\beta$, [17] conjectured that $\mathcal{S}_{D_\beta}$ is the Newton polytope of $\kappa_\beta$ and moreover that $\kappa_\beta$ are SNP; this is proved in [8]. Nonvanishing is also in P, provable using results of Section 4 in a manner analogous to that used for the Schubert polynomials.

# 3 The Schubitope

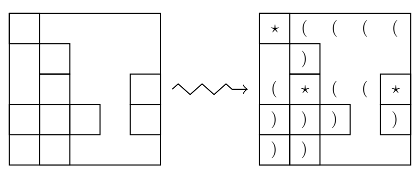
Fix $n \in \mathbb{Z}_{>0}$ and let $D \subseteq [n]^2$. We call $D$ a *diagram* and visualize $D$ as a subset of an $n \times n$ grid of boxes, oriented so that $(r, c) \in [n]^2$ represents the box in the $r$th row from the top and the $c$th column from the left. Given $S \subseteq [n]$ and a column $c \in [n]$, construct a string denoted $\mathsf{word}_{c,S}(D)$ by reading column $c$ from top to bottom and recording

- ( if $(r, c) \notin D$ and $r \in S$,
- ) if $(r, c) \in D$ and $r \notin S$, and
- $\star$ if $(r, c) \in D$ and $r \in S$.

Let $\theta_D^c(S) = \#\{\star\text{'s in } \mathsf{word}_{c,S}(D)\} + \#\{\text{paired ()'s in } \mathsf{word}_{c,S}(D)\}$ and

$$\theta_D(S) = \sum_{c=1}^{n} \theta_D^c(S).$$

**Example 3.1.** *In the diagram $D$ below, we labelled the corresponding strings for $\mathsf{word}_{c,S}(D)$ for $S = \{1, 3\}$. For instance, we see $\mathsf{word}_{5,\{1,3\}}(D) = (\star)$.*



The *Schubitope* $\mathcal{S}_D$, as defined in [17], is the polytope

$$\left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0} : \alpha_1 + \cdots + \alpha_n = \#D \text{ and } \sum_{i \in S} \alpha_i \leq \theta_D(S) \text{ for all } S \subseteq [n] \right\}. \tag{3.1}$$

## 3.1 Characterizations via tableaux

A *tableau* of *shape* $D$ is a map

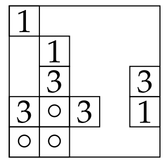
$$\tau : D \to [n] \cup \{\circ\},$$

where $\tau(r, c) = \circ$ indicates that the box $(r, c)$ is unlabelled. Let $\mathsf{Tab}(D)$ denote the set of such tableaux. One of the ideas in our proofs is to reformulate the original definition of $\theta_D(S)$ into the language of tableaux. Given $S \subseteq [n]$, define $\pi_{D,S} \in \mathsf{Tab}(D)$ by

$$\pi_{D,S}(r, c) := \begin{cases} r & \text{if } (r, c) \text{ contributes a "}\star\text{" to } \mathsf{word}_{c,S}(D), \\ s & \text{if } (r, c) \text{ contributes a ")" to } \mathsf{word}_{c,S}(D) \text{ which is} \\ & \text{paired with an "(" from } (s, c), \\ \circ & \text{otherwise.} \end{cases} \tag{3.2}$$

In (3.2) we pair by the standard "inside-out" convention.

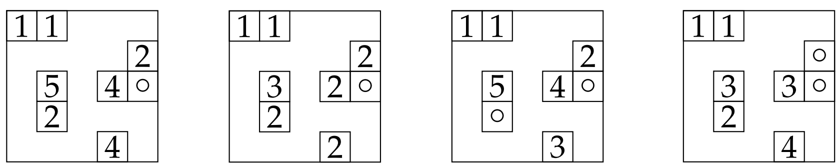
**Example 3.2.** *Continuing Example 3.1, below is* $\pi_{D,\{1,3\}}(D)$.



**Theorem 3.3.** *Let* $D \subseteq [n]^2$ *and* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ *with* $\alpha_1 + \cdots + \alpha_n = \#D$. *Then* $\alpha \in \mathcal{S}_D$ *if and only if for each* $S \subseteq [n]$, $\sum_{i \in S} \alpha_i \leq \#\pi^{-1}_{D,S}(S)$.

Define $\tau \in \mathsf{Tab}(D)$ to be *flagged* if $\tau(r,c) \leq r$ whenever $\tau(r,c) \neq \circ$. It is *column-injective* if $\tau(r,c) \neq \tau(r',c)$ whenever $r \neq r'$ and $\tau(r,c) \neq \circ$.

**Example 3.4.** *Of the tableaux of shape D below, only the second and fourth are flagged, and only the third and fourth are column-injective.*



Further, call a tableau $\tau \in \mathsf{Tab}(D)$ *perfect* if $\tau$ is flagged, column-injective, and if no boxes are left unlabelled, i.e., $\tau^{-1}(\{\circ\}) = \emptyset$. Say $\tau \in \mathsf{Tab}(D)$ has *content* $\alpha$ if $\#\tau^{-1}(\{i\}) = \alpha_i$ for each $i \in [n]$. Let $\mathsf{PerfectTab}(D, \alpha)$ denote the set of perfect tableaux of content $\alpha$.

We use Theorem 3.3 to prove:

**Theorem 3.5.** *Let* $D \subseteq [n]^2$ *and* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. *Then* $\alpha \in \mathcal{S}_D$ *if and only if* $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$.

# 4 Polytopal descriptions of perfect tableaux

By Theorem 3.5, to decide $\alpha \in \mathcal{S}_D$, it suffices to determine $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$. Thus it remains to analyze the complexity of deciding $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$.

For this, we construct a polytope that characterizes $\mathsf{PerfectTab}(D, \alpha)$. Given $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, define

$$\mathcal{P}(D, \alpha) \subseteq \mathbb{R}^{n^2}$$

to be the polytope with points of the form $(\alpha_{ij})_{i,j \in [n]} = (\alpha_{11}, \ldots, \alpha_{n1}, \ldots, \alpha_{1n}, \ldots, \alpha_{nn})$ governed by the inequalities (A)-(C) below.

(A) Column-Injectivity Conditions: For all $i, j \in [n]$,

$$0 \leq \alpha_{ij} \leq 1.$$

(B) Content Conditions: For all $i \in [n]$,

$$\sum_{j=1}^{n} \alpha_{ij} = \alpha_i.$$

(C) Flag Conditions: For all $s, j \in [n]$,

$$\sum_{i=1}^{s} \alpha_{ij} \geq \#\{(i,j) \in D : i \leq s\}.$$

**Theorem 4.1.** *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$.*

Theorem 4.1 formulates the problem of determining if $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ in terms of feasibility of an integer linear programming problem. In general, integral feasibility is $\mathsf{NP}$-complete. However, $\mathcal{P}(D, \alpha)$ is totally unimodular. Thus feasibility of the problem is equivalent to feasibility of its LP-relaxation:

**Theorem 4.2.** *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha_1 + \cdots + \alpha_n = \#D$. Then $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$ if and only if $\mathcal{P}(D, \alpha) \neq \emptyset$.*

By combining Theorems 3.5, 4.1, and 4.2 with the fact that $\mathcal{P}(D, \alpha)$ has a polynomial size halfspace description, it follows that $\alpha \in \mathcal{S}_D$ can be decided in $n^{O(1)}$-time. However, this result can be improved. If $D \subseteq [n]^2$ has many identical columns, then many of the flag conditions (C) will look essentially the same. Therefore, our final goal will be to construct a "compressed" version of $\mathcal{P}(D, \alpha)$ that removes some of the repetitive inequalities.

A tuple $\mathcal{C} = (m, \{P_k\}_{k=1}^{\ell})$ is a *compression* of $D \subseteq [n]^2$ if:

- $m \leq n$ is a nonnegative integer such that $(r, p) \notin D$ for $r > m$ and $p \in [n]$, and
- $P = P_1 \dot\cup \cdots \dot\cup P_\ell \subseteq [n]$ such that if $p, p' \in P_k$ then

  $$\{r \in [n] : (r, p) \in D\} = \{r \in [n] : (r, p') \in D\},$$

  and moreover if $D$ is nonempty in column $p$ then $p \in P_k$ for some $k \in [\ell]$.

For a compression $\mathcal{C}$ of $D \subseteq [n]^2$ and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{Z}^m_{\geq 0}$ define

$$\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha}) \subseteq \mathbb{R}^{m\ell} \tag{4.1}$$

to be the polytope with points of the form $(\tilde{\alpha}_{ik})_{i \in [m], k \in [\ell]}$ satisfying (A′)-(C′) below.

(A′) Column-Injectivity Conditions: For all $i \in [m], k \in [\ell]$,

$$0 \leq \tilde{\alpha}_{ik} \leq 1.$$

(B′) Content Conditions: For all $i \in [m]$,

$$\sum_{k=1}^{\ell} \#P_k \cdot \tilde{\alpha}_{ik} = \alpha_i.$$

(C′) Flag Conditions: For all $s \in [m], k \in [\ell]$,

$$\sum_{i=1}^{s} \tilde{\alpha}_{ik} \geq \#\{(i, p_k) \in D : i \leq s, p_k := \min P_k\}.$$

**Theorem 4.3.** *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) := (\alpha_1, \ldots, \alpha_m)$. Then $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathcal{P}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_m = \#D$, $\alpha_{m+1} = \cdots = \alpha_n = 0$, and $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha}) \neq \emptyset$.*

### 4.1 Deciding membership in the Schubitope

We use the above results to give a polynomial time algorithm to check if a lattice point is in the Schubitope. This more general result gives a polynomial time algorithm for any polynomial family whose Newton polytopes are Schubitopes. Let $D \subseteq [n]^2$, and fix a compression $\mathcal{C} = (m, \{P_k\}_{k=1}^{\ell})$ of $D$.

**Theorem 4.4.** *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\alpha \in \mathcal{S}_D$ if and only if $\alpha_1 + \cdots + \alpha_m = \#D$, $\alpha_{m+1} = \cdots = \alpha_n = 0$, and $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha}) \neq \emptyset$, where $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) := (\alpha_1, \ldots, \alpha_m)$.*

For each $k \in [\ell]$, let $p_k := \min P_k$ and $R_k(\mathcal{C}) := \{r \in [n] : (r, p_k) \in D\} \subseteq [m]$.

**Theorem 4.5.** *Given as input $\{R_k(\mathcal{C})\}_{k=1}^{\ell}$, $\{\#P_k\}_{k=1}^{\ell}$, and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{Z}^m_{\geq 0}$ satisfying $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_m = \#D$, one can decide if $\alpha := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, 0, \ldots, 0) \in \mathbb{Z}^n_{\geq 0}$ lies in $\mathcal{S}_D$ in polynomial time in $m$ and $\ell$.*

## 5 Application to $D(w)$: proof of Theorems 2.1 and 2.2

For the specialization to Rothe diagrams $D := D(w)$, the results of A. Fink-K. Mészáros-A. St. Dizier [8] imply

$$\alpha \in \mathcal{S}_{D(w)} \iff c_{\alpha,w} > 0.$$

Combining this with Theorem 3.5,

$$c_{\alpha,w} > 0 \iff \mathsf{PerfectTab}(D(w), \alpha) \neq \emptyset.$$

Further, if $\mathsf{PerfectTab}(D(w), \alpha) \neq \emptyset$, we can find $\tau \in \mathsf{PerfectTab}(D(w), \alpha)$ which is also strictly increasing along columns. Thus $\mathsf{PerfectTab}_{>}(D(w), \alpha) \neq \emptyset$, and Theorem 2.2 follows.

To obtain Theorem 2.1 we apply the results of Section 4 to $D(w)$. Suppose $\mathsf{code}(w) = (c_1, \ldots, c_L)$. Let $\sigma \in S_L$ be such that

$$w(\sigma(1)) < w(\sigma(2)) < \ldots < w(\sigma(L)).$$

Set $w(\sigma(0)) := 0$. The key lemma we need is:

**Lemma 5.1.** *For $1 \leq h \leq L$, and for all*

$$j_1, j_2 \in \{w(\sigma(h-1)) + 1, w(\sigma(h-1)) + 2, \ldots, w(\sigma(h)) - 1\},$$

*we have $(i, j_1) \in D(w)$ if and only if $(i, j_2) \in D(w)$.*

Using Lemma 5.1, there exists a compression $\mathcal{C} = (L, \{P_k\}_{k=1}^{\ell})$ of $D(w)$ where $\ell \leq 2L$. With the following statement, Theorem 4.5 proves Theorem 2.1.

**Proposition 5.2.** *There exists an $O(L^2)$-time algorithm to compute $\mathcal{C}, \{\#P_k\}_{k=1}^{\ell}$, and $\{R_k\}_{k=1}^{\ell}$ from the input $\mathsf{code}(w) = (c_1, \ldots, c_L)$.*

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