RHOMBIC TILINGS AND BOTT-SAMELSON VARIETIES

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ABSTRACT. S. Elnitsky (1997) gave an elegant bijection between rhombic tilings of $2n$-gons and commutation classes of reduced words in the symmetric group on n letters. P. Magyar (1998) found an important construction of the Bott-Samelson varieties introduced by H.C. Hansen (1973) and M. Demazure (1974). We explain a natural connection between S. Elnitsky's and P. Magyar's results. This suggests using tilings to encapsulate Bott-Samelson data (in type A). It also indicates a geometric perspective on S. Elnitsky's combinatorics. We also extend this construction by assigning desingularizations to the zonotopal tilings considered by B. Tenner (2006).

1. INTRODUCTION

Let $X = \mathsf{Flags}(\mathbb{C}^n)$ be the variety of complete flags $\mathbb{C}^0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$. The group $\mathsf{GL_n}(\mathbb{C})$ acts on the variety X by change of basis, as does its subgroup B of invertible upper triangular matrices and its maximal torus T of invertible diagonal matrices. The T-fixed points are in bijection with permutations w in the symmetric group $\mathfrak{S}_n$: they are the flags $F_\bullet^{(w)}$ defined by $F_k^{(w)} = \langle \vec{e}_{w(1)}, \vec{e}_{w(2)}, \ldots, \vec{e}_{w(k)} \rangle$ where $\vec{e}_i$ is the i-th standard basis vector. The **Schubert variety** X_w is the B-orbit closure of $F_\bullet^{(w)}$.

There is longstanding interest in singularities of Schubert varieties; see, for example, the text by S. Billey-V. Lakshmibai [BL00]. Famously, H.C. Hansen [Han73] and M. Demazure [Dem74] independently presented (in all Lie types) resolutions of singularities $BS^{(i_1, i_2, \ldots, i_{\ell(w)})}$ of X_w, one for each reduced word $s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ of w. M. Demazure called these resolutions **Bott-Samelson varieties** in reference to a related construction of R. Bott-H. Samelson [BS55]. In more recent work, P. Magyar [Mag98] found an important description of Bott-Samelson varieties.

We propose a canonical connection between P. Magyar's work and the rhombic tilings of S. Elnitsky [Eln97]. In this way, tilings graphically encapsulate Bott-Samelson data. (One should compare what follows to the similar use of X. Viennot's *heaps* [Vie89] to present Bott-Samelsons; see N. Perrin's [Per07] and B. Jones-A. Woo's [JW13].)

Given a permutation $w \in S_n$, the **Elnitsky $2n$-gon** $\mathsf{E}(w)$ has sides of length one, and these are labeled, in order, by $1, 2, \ldots, n, w(n), w(n-1), \ldots, w(1)$, in which the first n labels form half of a regular $2n$-gon, and sides with the same label are parallel. In Figure 1, we give the Elnitsky 14-gon for the permutation $7456312 \in S_7$; this example will be referenced throughout this work.

Let $\mathcal{T}(w)$ be the set of **rhombic tilings** of $\mathsf{E}(w)$ in which the rhombi have sides of length one and edges parallel to edges of $\mathsf{E}(w)$. The main result of S. Elnitsky's aforementioned

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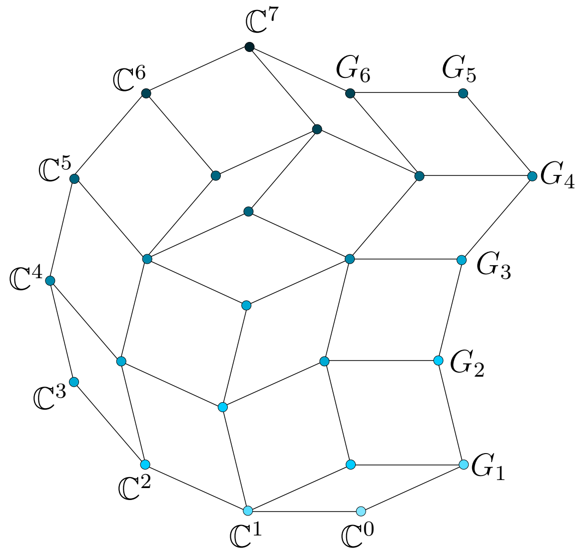


FIGURE 1. The rhombic tiling picture of Bott-Samelson varieties, for the polygon $\mathsf{E}(7456312)$.

work is that the set $\mathcal{T}(w)$ is in bijection with the commutation classes of reduced words of w [Eln97, Theorem 2.2].

We associate a vector space to each vertex of a tiling $T \in \mathcal{T}(w)$. Starting with the vertex between the edges labeled 1 and $w(1)$, label the vertices of $\mathsf{E}(w)$ in clockwise order by

$$\mathbb{C}^0, \mathbb{C}^1, \ldots, \mathbb{C}^n, G_{n-1}, G_{n-2}, \ldots, G_1.$$

In general, let V_x be the vector space associated to a vertex x in the tiling. The dimension of V_x is the minimal path length from $\mathbb{C}^0$ to x along tile edges. In Figure 1, we have only labeled the external vertices.

For adjacent vertices x and y in a tiling $T \in \mathcal{T}(w)$, write $x \to y$ if $\dim(V_x) + 1 = \dim(V_y)$. Let $\texttt{Vert}(T)$ be the vertices of T, and define

$$\mathcal{Z}_T := \left\{ (V_x : x \in \texttt{Vert}(T)) : V_y \subseteq V_z \text{ if } y \to z \right\} \subset \prod_{x \in \texttt{Vert}(T)} \mathsf{Gr}_{\dim(V_x)}(\mathbb{C}^n),$$

where $\mathsf{Gr}_k(\mathbb{C}^n)$ is the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$.

Define the map $\pi : \mathcal{Z}_T \to X$ by forgetting all vector spaces except those labeled by the vertices $G_1, G_2, \ldots, G_{n-1}$. In our example, π maps the point depicted in Figure 1 to the complete flag

$$\mathbb{C}^0 \subset G_1 \subset G_2 \subset G_3 \subset G_4 \subset G_5 \subset G_6 \subset \mathbb{C}^7.$$

The following theorem suggests a Schubert-geometric interpretation of tilings of Elnitsky polygons.

Theorem 1.1. *For $T \in \mathcal{T}(w)$, $\mathcal{Z}_T$ is a Bott-Samelson variety, i.e., a desingularization $\pi : \mathcal{Z}_T \to X_w$. Conversely, every Bott-Samelson variety $BS^{(i_1,\ldots,i_{\ell(w)})}$ is canonically isomorphic to $\mathcal{Z}_T$ for some $T \in \mathcal{T}(w)$ where $w = s_{i_1} \ldots s_{i_{\ell(w)}}$ and T is given in an explicit manner by* [Eln97, Theorem 2.2].

In Section 2, we prove Theorem 1.1. The remainder of this paper concerns other Bott-Samelson data encoded by tilings. In Section 3, we explain how the hexagon *flips* of

[Eln97, Section 3] may be interpreted geometrically. This naturally leads to the *zonotopal tilings* of [Ten06], each of which corresponds to a desingularization of a Schubert variety. We collect some additional discussion in Section 4; in particular, we explain how coloring rhombi of a tiling describes T-fixed points as well as a standard stratification of a Bott-Samelson variety.

2. Proof of Theorem 1.1

Two reduced words for w are **commutation equivalent** if they can be obtained from one another using only the relation $s_i s_j = s_j s_i$ when $|i-j|>1$.

By [Eln97, Theorem 2.2], the set $\mathcal{T}(w)$ bijects with commutation classes of reduced words of w. (Note that our orientation of the polygon is a horizontal reflection of the orientation given in [Eln97].) To link with [Mag98], we recall the bijection. Consider a tiling $T \in \mathcal{T}(w)$. The edges of T that coincide with edges of $\mathsf{E}(w)$ inherit the labels of those edges, and we label the interior edges of T so that parallel edges have the same labels.

Let B_0 be the **base boundary** of $\mathsf{E}(w)$, formed by the edges of the polygon appearing clockwise between $\mathbb{C}^0$ and $\mathbb{C}^n$. Pick any rhombus R_1 of T that shares two edges with B_0. Set $i_1 := d_1+1$, where d_1 is the minimum distance from $\mathbb{C}^0$ to R_1. Remove R_1 and define a new boundary, B_1, from B_0 by using the other two edges of R_1 instead. Now repeat this process: pick any rhombus R_2 that shares two edges with B_1; set $i_2 := d_2+1$, where d_2 is the minimum distance from $\mathbb{C}^0$ to R_2; remove R_2 and form a new boundary B_2. Iterating this process an additional $\ell(w)-2$ times produces $(i_1, i_2, \ldots, i_{\ell(w)})$, for which $s_{i_1}s_{i_2}\cdots s_{i_{\ell(w)}}$ represents a commutation class of reduced words for w. The other direction of the bijection is indicated below.

We now show that $\mathcal{Z}_T$ is isomorphic to $BS^{(i_1,i_2,\ldots,i_{\ell(w)})}$. P. Magyar [Mag98, Theorem 1] describes $BS^{(i_1,i_2,\ldots,i_{\ell(w)})}$ as a list $(F_\bullet^0,\ldots,F_\bullet^m)$ of $m+1$ flags where $F_\bullet^0$ is the base flag, and such that $F_\bullet^k$ agrees with $F_\bullet^{k-1}$ everywhere except possibly on the i_k-th subspace. Such a list of flags transparently corresponds in a one-to-one fashion to points in $\mathcal{Z}_T$: $F_\bullet^0$ is the base flag which is on the base boundary B_0 and in general, $F_\bullet^k$ is the flag on B_k.

Suppose that $\mathbf{j}=(j_1,j_2,\ldots)$ is commutation equivalent to $\mathbf{i}=(i_1,i_2,\ldots)$. It is well-known to experts that $BS^{\mathbf{i}}$ and $BS^{\mathbf{j}}$ are isomorphic varieties, but we include a proof for completeness. It suffices to prove this when $\mathbf{j}=(i_1,\ldots,i_{k+1},i_k,\ldots,i_{\ell(w)})$ differs from $\mathbf{i}$ only in positions k and $k+1$. The general result then follows by induction. Now, $(F_\bullet^0,\ldots,F_\bullet^m)$ is equivalent to a list of subspaces $(V_1,V_2,\ldots)$ satisfying:

- $\dim(V_k)=i_k$;
- $\mathbb{C}^{i_1-1}\subset V_1\subset\mathbb{C}^{i_1+1}$; that is, V_1 is contained in the (i_1+1)-dimensional subspace of $F_\bullet^0$ and contains the (i_1-1)-dimensional subspace of $F_\bullet^0$;
- V_2 is contained in the (i_2+1)-dimensional subspace of $F_\bullet^1$ and contains the (i_2-1)-dimensional subspace of $F_\bullet^1$; and so on.

Since $|i_{k+1}-i_k|>1$, the (i_k+1)-, (i_k-1)-, $(i_{k+1}+1)$-, and $(i_{k+1}-1)$-dimensional subspaces of $F_\bullet^k$ are precisely the subspaces of $F_\bullet^{k-1}$ with those dimensions. So if a generic element of $BS^{\mathbf{i}}$ is $(V_1,V_2,\ldots)$, then a generic element of $BS^{\mathbf{j}}$ is $(V_1,V_2,\ldots,V_{k+1},V_k,\ldots)$. That is, the isomorphism by switching factors:

$$\tau_k : \mathsf{Gr}_{i_1}(\mathbb{C}^n)\times\cdots\times\mathsf{Gr}_{i_k}(\mathbb{C}^n)\times\mathsf{Gr}_{i_{k+1}}(\mathbb{C}^n)\times\cdots\to\mathsf{Gr}_{i_1}(\mathbb{C}^n)\times\cdots\times\mathsf{Gr}_{i_{k+1}}(\mathbb{C}^n)\times\mathsf{Gr}_{i_k}(\mathbb{C}^n)\times\cdots$$

restricts to a canonical isomorphism from $BS^{(i_1,i_2,\dots)}$ to $BS^{(i_1,\dots,i_{k+1},i_k,\dots)}$. In other words, $\mathcal{T}(w)$ indexes Bott-Samelson varieties up to commutation equivalence.

Given $\mathbf{i} = (i_1, i_2, \dots)$ representing a commutation class for w (that is, $s_{i_1}s_{i_2}\cdots$ is a reduced decomposition of w), the inverse map to S. Elnitsky's bijection constructs an ordered tiling of $\mathsf{E}(w)$, as follows. For $k \geq 1$, set $w^{(k)} := s_{i_1}s_{i_2}\cdots s_{i_k}$. By [Eln97], for $1 \leq k \leq \ell(w)$, the values $w^{(k)}(i_k)$ and $w^{(k)}(i_k+1)$ label adjacent edges of the boundary B_{k-1}. Place a rhombus, R_k, so that two of its edges coincide with the edges labeled $w^{(k)}(i_k)$ and $w^{(k)}(i_k+1)$ in B_{k-1}, and define the new boundary B_k from B_{k-1} by using the other two edges of R_k. This explicitly picks $\mathcal{Z}_T$ from $\mathbf{i}$ such that $\mathcal{Z}_T \cong BS^{\mathbf{i}}$, as desired. □

Example 2.1. Consider the tiling $T \in \mathcal{T}(7456312)$ depicted in Figure 1. One way to select the rhombi $\{R_1, R_2, \dots\}$ described in the proof of Theorem 1.1 is shown in Figure 2, where we have recorded only the subscript k of the rhombus R_k. The labeling in this figure represents the commutation class of the reduced word

$$s_3s_4s_2s_5s_6s_5s_3s_4s_3s_2s_1s_5s_2s_3s_6s_4s_5$$

for the permutation 7456312. Any other such labeling of these tiles would produce a different, but commutation equivalent, reduced word. For example, the labeling obtained by swapping the selections for R_{14} and R_{15}, both of which share two edges with the boundary B_{15}, as indicated in Figure 2, produces the commutation equivalent reduced word

$$s_3s_4s_2s_5s_6s_5s_3s_4s_3s_2s_1s_5s_2s_6s_3s_4s_5.$$

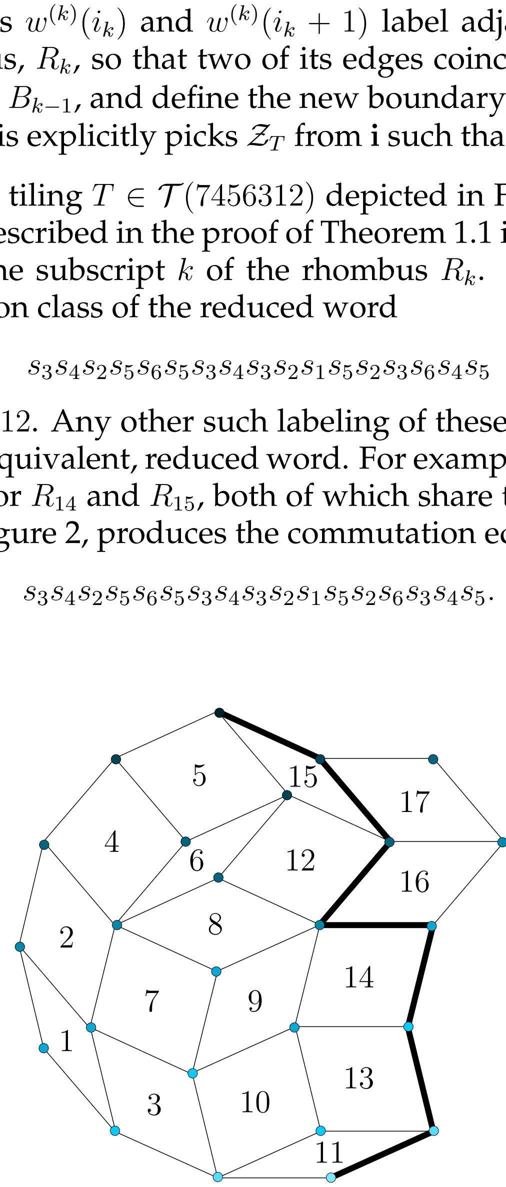


FIGURE 2. A labeling of the rhombi in an element of $\mathcal{T}(7456312)$, corresponding to the reduced word $s_3s_4s_2s_5s_6s_5s_3s_4s_3s_2s_1s_5s_2s_3s_6s_4s_5$ for the permutation 7456312. The boundary B_{15} is indicated by thick line segments.

3. FLIPS AND ZONOTOPAL TILINGS

3.1. **Flips.** Any pair of rhombic tilings of $\mathsf{E}(w)$ are connected by a sequence of hexagon "flips" [Eln97, Section 3]. The effect of a single flip is depicted in Figure 3.

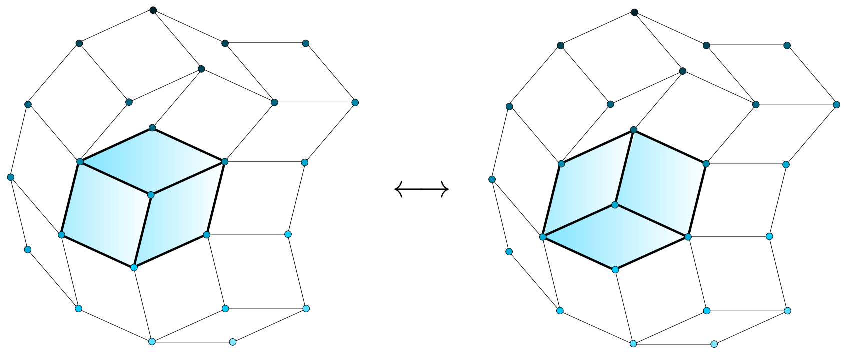


FIGURE 3. Two elements of $\mathcal{T}(7456312)$, related by a hexagon flip.

This flip has a geometric interpretation. Let $T, T' \in \mathcal{T}(w)$ be two rhombic tilings that differ by a single flip. Let T_H be the tiling of $\mathsf{E}(w)$ obtained from T (or, equivalently, from T') by erasing the three internal edges by which T and T' differ, and placing a hexagonal tile in the flip location. As before, associate vector spaces V_x to each vertex x in T_H, where $\dim(V_x)$ equals the distance from x to $\mathbb{C}^0$. The resulting space $\mathcal{Z}_{T_H}$ is similar to a Bott-Samelson variety: instead of being $\ell(w)$-fold iterated $\mathbb{CP}^1$-bundles over the base flag, we replace three of these $\mathbb{CP}^1$-bundles (corresponding to either triple of rhombi in the hexagon) by a $\mathsf{Flags}(\mathbb{C}^3)$-bundle. We then have

$$\begin{array}{ccc} \mathcal{Z}_T & & \mathcal{Z}_{T'} \\ & {\scriptstyle f_T}\searrow \quad \swarrow{\scriptstyle f_{T'}} & \\ & \mathcal{Z}_{T_H} & \end{array}$$

where the two maps are the projections determined by forgetting the vector space attached to the internal vertex of the hexagon.

3.2. **Zonotopal tilings.** The tiling T_H described above is a special case of the "zonotopal" tilings of Elnitsky polygons, which were studied by the third author in [Ten06]. To be precise, a **2-zonotope** is the projection of a regular q-dimensional cube onto the (2-dimensional) plane; equivalently, a 2-zonotope is a centrally symmetric convex polygon. A **zonotopal tiling** of a region is a tiling by 2-zonotopes. Figure 4 shows a zonotopal tiling of $\mathsf{E}(87465312)$ using one octagon, three hexagons, and ten rhombi.

Let $\mathcal{T}_{zono}(w)$ be the collection of zonotopal tilings of $\mathsf{E}(w)$, in which the tiles (2-zonotopes) have sides of length one and edges parallel to edges of $\mathsf{E}(w)$. Because rhombi are a type of 2-zonotope, we have $\mathcal{T}(w) \subseteq \mathcal{T}_{zono}(w)$.

Given a zonotopal tiling $Z \in \mathcal{T}_{zono}(w)$, we can define its corresponding **generalized Bott-Samelson variety** $\mathcal{Z}_Z$ by extending the construction from Section 3.1. For each vertex x in the zonotopal tiling, associate a vector space V_x whose dimension is the minimal path length from the bottom vertex to x along tile edges. Define

$$\mathcal{Z}_Z := \{(V_x : x \in \mathtt{Vert}(Z)) : V_y \subseteq V_z \text{ if } y \to z\}.$$

Let T be a rhombic tiling that refines Z; $\mathcal{Z}_T$ may be constructed as iterated $\mathbb{CP}^1$-bundles over a point. In the analogous construction of $\mathcal{Z}_Z$, for each $2k$-gon of Z, we replace k

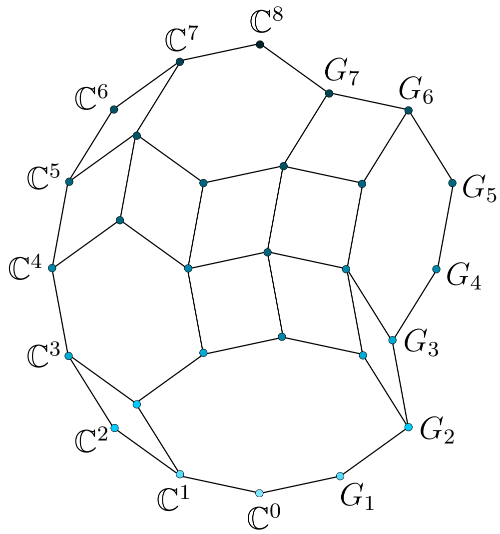


FIGURE 4. A zonotopal tiling for the permutation 87465312.

$\mathbb{CP}^1$-bundles with a $\mathsf{Flags}(\mathbb{C}^k)$-bundle. The variety $\mathcal{Z}_Z$ is smooth of dimension $\ell(w)$. Define $\pi_Z : \mathcal{Z}_Z \to X_w$ by forgetting all vector spaces except those labeled by the vertices $G_1, G_2, \ldots, G_{n-1}$.

Theorem 3.1. *Given a zonotopal tiling $Z \in \mathcal{T}_{zono}(w)$, its corresponding generalized Bott-Samelson variety $\mathcal{Z}_Z$ together with the map $\pi_Z : \mathcal{Z}_Z \to X_w$ is a resolution of singularities.*

Proof. Let $\pi_T : \mathcal{Z}_T \to X_w$ be a Bott-Samelson resolution where T is any rhombic tiling that refines Z. We know that π_T is birational, so let π'_T be its rational inverse. Let $f : \mathcal{Z}_T \twoheadrightarrow \mathcal{Z}_Z$ be the projection determined by forgetting the vector spaces attached to the internal vertices of the $2k$-gons. Since f is surjective, the image of π_Z is indeed X_w and the following commutative diagram implies that $f \circ \pi'_T$ is a rational inverse to π_Z.

$$\begin{array}{ccc} \mathcal{Z}_T & \xrightarrow{f} & \mathcal{Z}_Z \\ & \searrow^{\pi_T} & \downarrow^{\pi_Z} \\ & & X_w \end{array}$$

It follows that $\pi_Z : \mathcal{Z}_Z \to X_w$ is also a resolution of singularities. □

The zonotopal tilings $\mathcal{T}_{zono}(w)$ of $\mathsf{E}(w)$ have a natural poset structure, as studied by the third author in [Ten06]. The order relation in this poset is given by reverse edge inclusion. Thus the rhombic tilings are the minimal elements in the poset. A pair of rhombic tilings differ by a single hexagon flip if and only if they are covered by a common element. Similarly, one can get a broader sense of how closely two rhombic tilings (equivalently, two commutation classes of reduced words for w) are related by determining their least upper bound in this poset. Geometrically, the relations in the poset $\mathcal{T}_{zono}(w)$ correspond to the projections $\mathcal{Z}_Z \twoheadrightarrow \mathcal{Z}_{Z'}$ between two generalized Bott-Samelsons for X_w.

By [Ten06, Theorem 6.13], the poset of zonotopal tilings of $\mathsf{E}(w)$ has a unique maximal element $\hat{Z}$ exactly in the case that w avoids the patterns 4231, 4312, and 3421. In this case, there is a distinguished $\mathcal{Z}_{\hat{Z}}$ with a projection $\mathcal{Z}_Z \twoheadrightarrow \mathcal{Z}_{\hat{Z}}$ from every other generalized Bott-Samelson. Such permutations have been enumerated by T. Mansour [Man06].

For comparison, consider Elnitsky polygons whose zonotopal tilings do not contain any hexagonal tiles (equivalently, those polygons with a unique zonotopal tiling). These correspond to 321-avoiding permutations, which are exactly those whose reduced words contain no long braid moves [BJS93, Theorem 2.1] (see also [Ten15, Section 3] for more general results relating pattern avoidance and reduced words). The unique tiling in this case is a deformation of the skew shape associated to the permutation by considering its *Rothe diagram* and removing empty rows and columns. A standard filling orders the tilings in the sense of [Eln97] (and the final paragraph of the proof of Theorem 1.1).

We now have the following result (cf. [Ele15, Remark 3.1], where this fact for ordinary Bott-Samelsons is noted).

Proposition 3.2. *Suppose that $Z \in \mathcal{T}_{zono}(w)$, and that the number of $2i$-sided tiles in Z is t_i, for each $i \geq 1$. Then the Poincaré polynomial of the cohomology ring $H^\star(\mathcal{Z}_Z)$ is*

$$\sum_{k=0}^{\ell(w)} \dim H^{2k}(\mathcal{Z}_Z)q^k = \prod_{i\geq 1}[i]_q!^{t_i},$$

where $[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$ and $[i]_q! := [i]_q[i-1]_q \cdots [1]_q$.

Proof. The variety $\mathcal{Z}_Z$ is constructed as iterated flag bundles over a point, where t_i of the fibrations are by $\mathsf{Flags}(\mathbb{C}^i)$. It is a standard fact (following from the Schubert decomposition of $\mathsf{Flags}(\mathbb{C}^i)$) that the Poincaré polynomial of $H^\star(\mathsf{Flags}(\mathbb{C}^i))$ is $[i]_q!$ (indeed, $[i]_q!$ is the ordinary generating function for $\mathfrak{S}_i$ with each permutation weighted by Coxeter length). The proposition now follows from the Leray-Hirsch theorem (cf. [Hat02, Theorem 4D.1]). □

4. Additional discussion

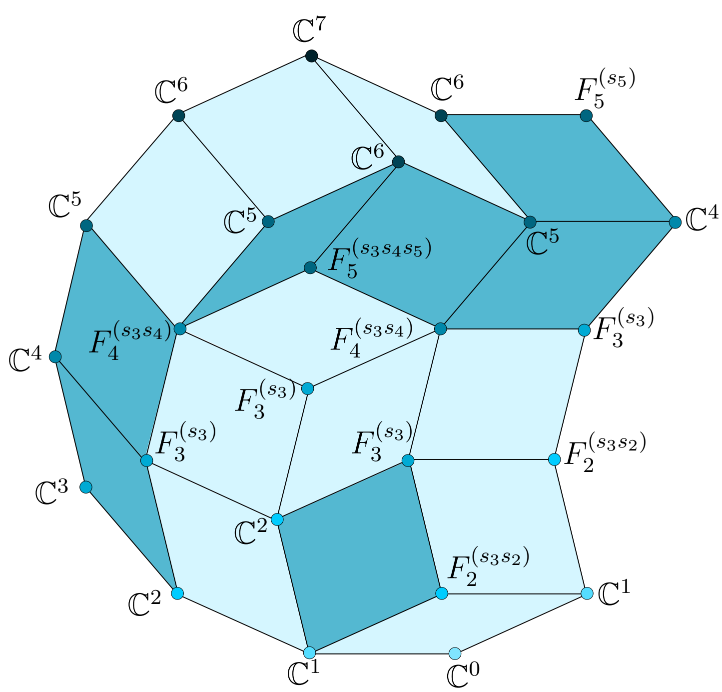


FIGURE 5. A coloring corresponding to a fixed point of $\mathcal{Z}_T$.

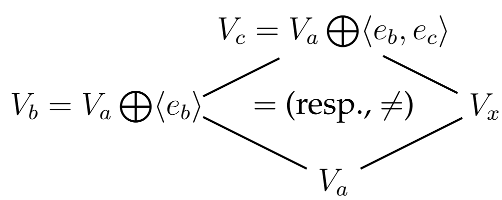
One may reformulate certain results about $BS^{\mathbf{i}}$ in terms of rhombic colorings; we refer to [Esc16, Section 3.2] for background with further references.

Proposition 4.1. *For $T \in \mathcal{T}(w)$, the T-fixed points of $\mathcal{Z}_T$ (under the diagonal action) are in one-to-one correspondence with bipartitions of the rhombi of T.*

Proof. Consider a 2-coloring of the rhombi of T representing the bipartition (as shown in Figure 5). There is a unique way to choose $\{V_x\}_{x \in \mathtt{Vert}(T)}$ such that

(1) each V_x is the span of a subset of the standard basis; and,
(2) for any rhombus, its two vector spaces of common dimension are the same (resp., different) if the rhombus is light-colored (resp., dark-colored).

Since the T-action is diagonal, if $\{V_x\}_{x \in \mathtt{Vert}(T)}$ is a T-fixed point of $\mathcal{Z}_T$, then each V_x must be T-fixed, i.e., each V_x must be spanned by a subset of the standard basis $\{e_1, \ldots, e_n\}$. Using the required containment relations, we can inductively determine V_x for each vertex of T by following an ordering of the rhombi given by a representative of the commutation class of T. At a particular colored rhombus, we make the two vector spaces of common dimension the same (resp., different) if the rhombus is light-colored (resp., dark-colored).



Conversely, every T-fixed point can be indicated by such a coloring. □

M. Demazure [Dem74] used the T-fixed points to prove that the image of $BS^{(i_1, i_2, \ldots)}$ under the Bott-Samelson map π is indeed the Schubert variety $X_{s_{i_1} s_{i_2} \cdots}$. These fixed points are also useful in the study of moment polytopes of Bott-Samelson varieties, and for other applications.

These colorings also correspond to a stratification of $\mathcal{Z}_T$ by smaller Bott-Samelsons. Given a coloring, the corresponding stratum has the property that, for any light-colored rhombus, its two vector spaces of common dimension are equal. The dark-colored rhombi impose no conditions. The unique smallest stratum corresponds to the all-light coloring, whereas the unique largest stratum corresponds to the all-dark one. For background, see [Esc16, Section 4.3]

In [Eln97], the author extends his main construction to the other Weyl groups of classical Lie type. This seems related to the Bott-Samelsons for the associated Lie groups.

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