SYMMETRIC GROUP REPRESENTATIONS AND $\mathbb{Z}$

ANSHUL ADVE AND ALEXANDER YONG

Let $\mathfrak{S}_n$ be the symmetric group of permutations of $\{1, 2, \ldots, n\}$. A *representation* is a homomorphism $\rho : \mathfrak{S}_n \to \mathsf{GL}(V)$ where V is a vector space over $\mathbb{C}$. Equivalently, V is an $\mathfrak{S}_n$-module under the action defined by $\sigma \cdot v = \rho(\sigma)v$, for $\sigma \in \mathfrak{S}_n$ and $v \in V$. Then ρ is *irreducible* if there is no proper $\mathfrak{S}_n$-submodule of V. Conjugacy classes and hence irreducible representations of $\mathfrak{S}_n$ biject with $\mathsf{Par}(n)$, the partitions of size n.

Consider three families of numbers from the theory:

(I) The *character* of ρ is

$$\chi^\rho : \mathfrak{S}_n \to \mathbb{C}; \quad \sigma \mapsto \mathrm{tr}(\rho(\sigma)).$$

Textbooks focus on the case $V = V_\lambda$ is irreducible (because of Maschke's theorem). Since characters are constant on each conjugacy class μ, one needs only $\chi^\lambda(\mu)$. These are computed by the Murnaghan-Nakayama rule (see below). More recent results include bounds on (normalized) character evaluations [Ro96, FePi11].

(II) If V_λ and V_μ are irreducible $\mathfrak{S}_m$ and $\mathfrak{S}_n$-modules, respectively, then $V_\lambda \otimes V_\mu$ is an irreducible $\mathfrak{S}_m \times \mathfrak{S}_n$-module. If V_ν is an irreducible $\mathfrak{S}_{m+n}$-representation, it restricts to a $\mathfrak{S}_m \times \mathfrak{S}_n$-representation $V_\nu \downarrow^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n}$. The *Littlewood-Richardson coefficient* is

$$c^\nu_{\lambda,\mu} = \text{multiplicity of } V_\lambda \otimes V_\mu \text{ in } V_\nu \downarrow^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n}.$$

Many *Littlewood-Richardson rules* are available to count $c^\nu_{\lambda,\mu}$ [St99].

(III) If V_λ, V_μ are $\mathfrak{S}_n$-modules then so is $V_\lambda \otimes V_\mu$. Hence we may write

$$V_\lambda \otimes V_\mu \cong \bigoplus_{\nu \in \mathsf{Par}(n)} V_\nu^{\oplus g_{\lambda,\mu,\nu}}.$$

Here, $g_{\lambda,\mu,\nu}$ is the *Kronecker coefficient*. One has an $\mathfrak{S}_3$-symmetric but cancellative formula $g_{\lambda,\mu,\nu} = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma)$; it is an old open problem to give a manifestly nonnegative combinatorial rule. The study of Kronecker coefficients has been given new impetus from *Geometric Complexity Theory*, an approach to the P vs NP problem; see [BlMuSo15] and the references therein.

This note visits a rudimentary point. While for finite groups, character evaluations are algebraic integers, for $\mathfrak{S}_n$, in fact $\chi^\lambda(\mu) \in \mathbb{Z}$. Moreover, by definition, $c^\nu_{\lambda,\mu}, g_{\lambda,\mu,\nu} \in \mathbb{Z}_{\geq 0}$. We remark the three converses hold.[1] The proof uses standard facts, but we are unaware of any specific reference in the textbooks [Ja78, FuHa99, St99, Sa01], or elsewhere.

Theorem. *Every integer is infinitely often an irreducible $\mathfrak{S}_n$-character evaluation. Every non-negative integer is infinitely often a Littlewood-Richardson coefficient, and a Kronecker coefficient.*

Corollary A. *There exists a value-preserving multiset bijection between the Littlewood-Richardson and Kronecker coefficients.*

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[1]inspired by P. Polo [Po99]: every $f \in 1 + q\mathbb{Z}_{\geq 0}[q]$ is a Kazhdan-Lusztig polynomial for some $\mathfrak{S}_n$

Proof. Clearly, the Theorem implies that for each $k \in \mathbb{Z}_{\geq 0}$, the sets

$$\mathsf{LR}_k = \{(\lambda, \mu, \nu) : c_{\lambda,\mu}^{\nu} = k\} \text{ and } \mathsf{Kron}_k = \{(\lambda, \mu, \nu) : g_{\lambda,\mu,\nu} = k\}$$

are countably infinite and thus in bijection. □

Desirable would be a construction of an injection $\mathsf{Kron}_k \hookrightarrow \mathsf{LR}_k$ for each $k \in \mathbb{Z}_{\geq 0}$ (avoiding the countable axiom of choice). That should solve the Kronecker problem in (III), by reduction to (II). This we cannot do. However, there has been success in this vein [KnMiSh04] on another counting problem. See the Remark at the end of this paper.[2]

Proof of the Theorem: The *Murnaghan-Nakayama rule* states $\chi^{\lambda}(\mu) = \sum_T (-1)^{\mathrm{ht}(T)}$, where T is a tableaux of shape λ with μ_i many labels i, the entries are weakly increasing along rows and columns, and the labels i form a connected skew shape T_i with no 2×2 subsquare; $\mathrm{ht}(T)$ is the sum of the heights of each T_i, i.e., one less than the number of rows of T_i.

We sharpen the assertion about $\chi^{\lambda}(\mu)$. In particular, for a given n, we consider the intervals of consecutive integers achievable as character evaluations for $\mathfrak{S}_n$. From the rule, the character of the *defining representation* satisfies $\chi^{(n-1,1)}(\mu) = \#(1\text{'s in } \mu) - 1$ (see also [Ja78, Lemma 6.9]). Hence, $\chi^{(n-1,1)}$ takes the values $[0, n-2]$. Similarly, $\chi^{(2,1^{n-2})}$ achieves an interval of negative integers: Take $k \in [1, n-5] \cup \{n-3\}$. If $k \not\equiv n \mod 2$, let $\mu = (n-k-1, 1^{k+1})$. Otherwise, if $k \equiv n \bmod 2$, let $\mu = (n-k-4, 3, 1^{k+1})$. Note that if $k = n-6$, let μ be these parts in decreasing order. In either case, the rule shows $\chi^{(2,1^{n-2})}(\mu) = -k$. Thus, for $n \geq 5$, $[-(n-5), n-2] \subseteq \{\chi^{\lambda}(\mu) : \lambda, \mu \in \mathsf{Par}(n)\}$. Taking $n \to \infty$ implies the statement regarding character evaluations.

The *Kostka coefficient* $K_{\lambda,\mu}$ is the number of *semistandard Young tableaux* of shape λ with content μ, i.e., fillings of λ with μ_i many i's such that rows are weakly increasing and columns are strictly increasing.

Lemma. *Every nonnegative integer is infinitely often a Kostka coefficient.*

Proof. Clearly, $K_{(1+j,1^{k-1}),(j,1^k)} = k$ for $j \geq 1$. The lemma then follows. □

The Littlewood-Richardson coefficient claim holds since it is long known that Kostka coefficients are a special case. To be specific, $K_{\lambda,\mu} = c_{\sigma,\lambda}^{\tau}$ where

$$\tau_i = \mu_i + \mu_{i+1} + \cdots, \ i = 1, 2, \ldots, \ell(\mu), \text{ and}$$

$$\sigma_i = \mu_{i+1} + \mu_{i+2} + \cdots, \ i = 1, 2, \ldots, \ell(\mu) - 1.$$[3]

For $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $\lambda[N] := (N - |\lambda|, \lambda_1, \lambda_2, \ldots)$. F. D. Murnaghan [Mu38] proved that for an integer $N \gg 0$, $\chi^{\lambda[N]} \otimes \chi^{\mu[N]} = \sum_{\nu} \overline{g_{\lambda,\mu,\nu}} \chi^{\nu[N]}$. The $\overline{g_{\lambda,\mu,\nu}}$ are called *stable Kronecker coefficients* and are evidently a special case of Kronecker coefficients. When $|\lambda| + |\mu| = |\nu|$ one has $\overline{g_{\lambda,\mu,\nu}} = c_{\lambda,\mu}^{\nu}$. Hence one infers the Kronecker coefficient assertion. □

[2]There is debate about the idiomatic meaning of *counting rule* or *manifestly nonnegative combinatorial rule* etc. Consider the (adjusted) Fibonacci numbers $(1, 1, 2, 3, 5, 8, 13, \ldots)$. A counting rule is that F_n counts the number of $(1, 2)$-lists whose sum is n. The recursive (and computationally efficient) description is $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ where $F_0 = F_1 = 1$. Construct a binary tree $\mathcal{T}_n$ with root labelled F_n; each node of label F_i has a left child F_{i-1} and right child F_{i-2}. Leaves of $\mathcal{T}_n$ are labelled F_1 or F_0. F_n counts the number of leaves of $\mathcal{T}_n$. The latter description restates the recurrence and is not, *per se*, a counting rule.

[3]This reduction is used by H. Narayanan [Na06] to show computing $c_{\lambda,\mu}^{\nu}$ is a #P problem.

When, e.g., $n = 25$, all of $[-853, 949]$ appear as some $\chi^\lambda(\mu)$, but the proof merely guarantees $[-20, 23]$. Let ℓ_n be the maximum size of an interval of consecutive character evaluations for $\mathfrak{S}_n$. Trivially, the results of [Ro96, FePi11] imply upper bounds for ℓ_n. Can one prove better upper or lower bounds for ℓ_n?

Let A_n be the *alternating group* of even permutations in $\mathfrak{S}_n$. Sources about the representation theory of A_n include [JaKe09, Section 2.5] and [FuHa99, Section 5.1]. Character evaluations of A_n are not always integral, however:

Corollary B. *Every integer appears infinitely often as an A_n-irreducible character evaluation.*

Proof. Let $\psi^\lambda = \chi^\lambda \downarrow_{\mathsf{A}_n}^{\mathfrak{S}_n}$ be the character of the restriction of the $\mathfrak{S}_n$-irreducible V_λ. If μ is not a partition with distinct odd parts then the conjugacy class in $\mathfrak{S}_n$ of cycle type μ is also a conjugacy class of A_n. If λ is not a self-conjugate partition, the restriction is an A_n-irreducible and also $\psi^\lambda(\mu) = \chi^\lambda(\mu)$. Repeat the Theorem's character argument, since for $n \geq 4$ neither λ used is self-conjugate, and since for $k \geq 1$, μ has equal parts. ☐

Definition. For a countable indexing set A, a family of nonnegative integers $(a_\alpha)_{\alpha \in \mathsf{A}}$ is *entire* if every $k \in \mathbb{Z}_{\geq 0}$ appears infinitely often.

Many of the nonnegative integers arising in algebraic combinatorics are entire. For example, this is true for the theory of *Schubert polynomials* (we refer to [Ma01] for references). If $w_0 \in \mathfrak{S}_n$ is the longest permutation then $\mathfrak{S}_{w_0}(x_1, \ldots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. If $w \neq w_0$, $w(i) < w(i+1)$ for some i. Then $\mathfrak{S}_w(x_1, \ldots x_n) = \partial_i \mathfrak{S}_{ws_i}(x_1, \ldots, x_n)$ where $\partial_i = \frac{f - s_i(f)}{x_i - x_{i+1}}$ and s_i is the simple transposition interchanging $i, i+1$. Nontrivially, each $\mathfrak{S}_w \in \mathbb{Z}_{\geq 0}[x_1, x_2, \ldots]$. Moreover, $\mathfrak{S}_w = \mathfrak{S}_{w \times 1}$ where $w \times 1 \in \mathfrak{S}_{n+1}$ is the usual image of $w \in \mathfrak{S}_n$. Thus we can discuss $\mathfrak{S}_w$ for $w \in \mathfrak{S}_\infty$; these form a $\mathbb{Z}$-linear basis of $\mathbb{Z}[x_1, x_2, \ldots]$. The *Schubert structure constants* $C_{u,v}^w := [\mathfrak{S}_w]\mathfrak{S}_u \mathfrak{S}_v \in \mathbb{Z}_{\geq 0}$ for geometric reasons. The *Stanley symmetric function* is defined by $F_w = \lim_{m \to \infty} \mathfrak{S}_{1^m \times w} \in \mathbb{Z}[[x_1, x_2, \ldots]]$; here $1^m \times w \in \mathfrak{S}_{m+n}$ sets $1^m \times w(i)$ equal to i if $1 \leq i \leq m$ and equal to $w(i - m + 1) + m$ otherwise. F_w is Schur-nonnegative.

Corollary C. *These families of nonnegative integers are entire:*

(a) *The coefficients of monomials in Schubert polynomials.*
(b) *The Schubert structure constants.*
(c) *The coefficients of Schur functions in Stanley symmetric functions.*

Proof. (a) is true by the Lemma since when w is *Grassmannian* (has at most one descent), $\mathfrak{S}_w(x_1, \ldots, x_n)$ is a Schur polynomial s_λ. When u, v and w are Grassmannian with descent position d, then $C_{u,v}^w$ is a Littlewood-Richardson coefficient so the Theorem implies (b). Finally, when w is 321-avoiding (i.e., there does not exist indices $i < j < k$ such that $w(i) > w(j) > w(k)$), $F_w = s_{\nu/\lambda} = \sum_\mu c_{\lambda,\mu}^\nu s_\mu$ is a skew Schur function. Hence, here the coefficient (c) is $c_{\lambda,\mu}^\nu$ and we apply the Theorem. ☐

Abstractly, all entire families are mutually in value-preserving bijection. However, for Corollary C one can say more: (a) and (c) are a special cases of (b) (see [BeSo98] and [BuSoYo05]). Can one construct a "wrong way map" (as in $\mathbb{Q} \hookrightarrow \mathbb{N}$) for either (b)$\hookrightarrow$(a) or (b)$\hookrightarrow$(c) (thereby finding a rule for $C_{u,v}^w$)? A special case indicating the difficulty is:

Problem. *Construct an explicit value-preserving injection between Littlewood-Richardson and Kostka coefficients.*

Remark. Finding a wrong way map has solved a significant counting rule problem concerning A. Buch-W. Fulton's *quiver coefficients*. These arise in the study of degeneracy loci of vector bundles over a smooth projective algebraic variety. It was conjectured by those two authors that these integers are nonnegative, with a conjectural counting rule. Also, A. Buch showed that special cases of the quiver coefficients are the numbers from (c) above. The resolution of this problem, due to A. Knutson-E. Miller-M. Shimozono, came by establishing the *opposite*: quiver coefficients are special cases of the well-understood numbers (c). We refer to the solution [KnMiSh04] for background and references.

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UNIVERSITY LABORATORY HIGH SCHOOL, URBANA, IL 61801

E-mail address: anshul.adve@gmail.com

DEPT. OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801

E-mail address: ayong@uiuc.edu